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PART I

SYMMETRY GROUPS AND ALGEBRAS

1 Introduction

Symmetry principles and geometrical/topological concepts are central to many of the most interesting developments in modern physics. Sophisticated mathematical advances in applications of groups, algebras, geometry, and topology to physical systems are now in routine use by theoretical physicists, and symmetry principles and concepts pervade the language that we use in our physical descriptions; but this was not always so.

Groups were invented by mathematicians who rejoiced that they had finally discovered something that was of no practical use to the natural scientists [72].[1] Lie algebras were invented largely by the Norwegian mathematician Sophus Lie and the German mathematician Wilhelm Killing in the period ~1873–1890. Just as differential geometry was developed by mathematicians more than a half century before Einstein (with the help of his mathematician friend Marcel Grossmann) adapted it to the description of gravity in his theory of general relativity, Lie algebras and Lie groups were developed mathematically decades before they began to find serious applications in physics. Likewise, just as the development of general relativity was the impetus for the introduction of differential geometry into physics applications in the period 1912–1915, the realization that Lie algebras and Lie groups might be important for physics was largely an outgrowth of the invention of quantum mechanics in 1925–1926.

In general relativity it was Einstein's realization that gravity must be associated with curved spacetime that necessitated a mathematical description of curvature that was intrinsic to the four-dimensional spacetime manifold. That description (differential geometry) had been invented by Riemann (building on the work of Gauss) more than half a century before. In quantum theory, sets of operators that close under commutation belong to a finite-dimensional Lie algebra and transformations are described by a finite number of continuous parameters belonging to a Lie group. Thus, Lie algebras and Lie groups provided a natural language to describe physical problems in the new quantum mechanics. However, many were slow to grasp this insight. Eugene Wigner, who would win a Nobel Prize for his applications of symmetry principles to nuclear and elementary particle physics, was advised more than once by giants of the field that the use of group

[1] Not the only time that mathematicians have erred in this expectation. Physicists have shown themselves to be quite clever in co-opting pure mathematics for their own purposes!

theory in quantum mechanics was a passing fad, best put aside in favor of less abstract approaches based on more traditional mathematics.[2]

Today symmetry concepts are integral to much of our thinking and discussion in many areas of physics, either explicitly or implicitly; so much so that it is now possible to teach angular momentum theory or the significance of conservation laws in quantum mechanics without much thought for the underlying Lie groups and Lie algebras that are ultimately responsible for these concepts. As a consequence, modern physics undergraduate students often gain a vague appreciation for, but generally do not get a systematic and mathematical grounding in, the role of symmetry in modern physics research.

Furthermore, in recent decades there has been an explosion of ideas built on geometrical and topological concepts in various fields of physics. Initially these concepts were pervasive in disciplines like elementary particle physics because they most often appeared as solutions of relativistic quantum field theories. But now geometrical and topological themes are proliferating in such unexpected research domains as materials science and condensed matter physics, which are built on non-relativistic quantum fields. These developments have significant implications for fundamental physics, but also may be of large importance for practical applications in quantum information, quantum computing, and related disciplines. Today, even experimentalists in elementary particle physics, broad ranges of condensed matter physics and material science, and quantum information and computing must at least know the language and concepts of topological quantum theories. Needless to say, most undergraduate physics students are even less prepared to deal with issues in topology and advanced geometry than they are to deal with those from sophisticated applications of groups and algebras in modern physics.

This book addresses the issues described above by providing a unified and pedagogical discussion of groups, algebras, geometry, and topology in modern physics. It is tailored specifically for physicists and students with at least an advanced undergraduate preparation, but without a specific background in these areas.

[2] Though of course the matrices employed in some formulations of quantum mechanics were themselves mathematics that had not been much used in physics prior to the advent of quantum theory (and that were, in fact, closely related to Lie groups). Schrödinger reputedly advised a young Wigner that the idea that Lie groups were relevant to quantum physics would be largely forgotten within five years. Not the first time that a scientist of considerable renown was in serious error about the future of a field!

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2 Some Properties of Groups

Our goal in this book is to examine basic principles of symmetry, topology, and geometry in the context of modern research in physics. We begin with symmetry and the mathematical concept of a *group*. In this chapter some fundamental definitions and terminology will be introduced, using as illustration a few simple groups that often have transparent geometrical or combinatorial interpretations.

2.1 Invariance and Conservation Laws

The essence of a symmetry principle is that some quantity is fundamentally unobservable and that this implies an invariance under a related mathematical transformation. For a geometrical object, this may be illustrated in terms of its (geometrical) *covering operations*: the set of (1) rotations, (2) inversions, and (3) reflections that leave it indistinguishable from the object before the transformation. For example, a sphere has a high degree of symmetry and it looks the same after any rotation, reflection, or inversion. A square has lower symmetry, but it is unchanged after rotation by $\frac{\pi}{2}$ or after various reflections, and so on.

This fundamental symmetry principle also may be implemented in more abstract terms. In particular, we may consider mathematical operations that are not essentially geometrical in nature such as the various transformations important in quantum mechanics. Suppose that a symmetry operation is implemented by the quantum-mechanical operator U. If the Hamiltonian operator H is invariant under this symmetry transformation,[1]

$$H = UHU^{-1}, \tag{2.1}$$

where the inverse U^{-1} is defined through $UU^{-1} = 1$. Operating from the right with U,

$$HU = UHU^{-1}U = UH \quad \rightarrow \quad [U, H] = 0, \tag{2.2}$$

[1] In quantum theory operators are often distinguished from non-operator quantities by the use of a different font or by a special notation such as placing a hat over operators: $\hat{H}$. In the interest of clean and concise notation, we will let the context dictate which quantities are operators and not use a separate notation for them except in special circumstances.

Table 2.1. Some symmetries and associated conservation laws

Non-observable	Transformation	Conservation law
Absolute position	Space translation: $x \to x + \delta x$	Momentum
Absolute time	Time translation: $t \to t + \delta t$	Energy
Absolute direction	Rotation: $\theta \to \theta + \delta\theta$	Angular momentum

where the *commutator* of two operators A and B is defined by

$$[A, B] \equiv AB - BA. \tag{2.3}$$

But in quantum mechanics the Heisenberg equation of motion for an operator U is

$$\frac{dU}{dt} = \frac{\partial U}{\partial t} + \frac{i}{\hbar}[U, H], \tag{2.4}$$

and if U has no explicit time dependence it is a *constant of motion* if it commutes with H. For most (not all) cases the operators U may be chosen to be unitary and written

$$U = e^{i\alpha A}, \tag{2.5}$$

where α is a real number and the unitarity of U, defined by $U^\dagger = U^{-1}$, implies that A is hermitian ($A = A^\dagger$, where $A^\dagger$ indicates hermitian conjugation of A: in a matrix representation, complex conjugate each element and interchange rows and columns). But in quantum mechanics hermitian operators imply observables (their real eigenvalues) and we conclude that invariance of a Hamiltonian operator with respect to some unitary transformation leads to a conservation law since, if U is a constant of motion, A will be also. Generally, the eigenvalues of A will be conserved quantum numbers and the operator conjugate to A in the sense of the uncertainty principle will be associated with an inherently unobservable quantity. Table 2.1 displays the relations among some well-known symmetries, conservation laws, and corresponding non-observables, as illustrated in Examples 2.1 and 2.2.

Example 2.1 The impossibility of determining absolute spatial position (homogeneity of space) implies an invariance under spatial translations. Conservation of linear momentum follows directly from this invariance.

Example 2.2 Inability to distinguish absolute orientation (isotropy of space) implies an invariance under rotations, which leads directly to conservation of angular momentum.

The natural framework for analysis of such symmetries is that of group theory and the algebras of its associated operators. Let us now turn to a consideration of those mathematical techniques, with groups to be introduced in this chapter and algebras in Ch. 3.

2.2 Definition of a Group

A *set* is a collection of distinct objects having the properties reviewed in Box 2.1. A *group* is a set $G = \{x, y, \ldots\}$ for which a binary operation $a \cdot b$ called *group multiplication* (or, more mathematically, *composition*) is defined, which has the following properties.

1. *Associativity:* Multiplication is associative, $(x \cdot y) \cdot z = x \cdot (y \cdot z)$ for each x, y, and z in G.
2. *Closure:* If x and y are elements of G, then $x \cdot y$ is an element of G also.
3. *Unique identity:* An identity element e exists such that $e \cdot x = x \cdot e = x$ for any x in G.
4. *Unique inverse:* Each group element x has an inverse x^{-1} such that $xx^{-1} = x^{-1}x = e$.

Box 2.1 Properties of Sets

A *set* is a collection of distinct objects considered as an entity. The members of a set are called *elements* and they can be almost anything (including other sets). The number of members is called the *cardinality* of the set.

Defining elements of a set: A set may be populated with elements in two ways.

1. List the elements explicitly: $A = \{\text{red}, \text{ yellow}, \text{ green}\}$.
2. Specify a rule giving all elements: "the negative integers greater than −4." A common rule is to specify a property $P(x)$ that all members x of a set have using *set-builder notation*: $\{x|P(x)\}$ means "the set of all x such that the property $P(x)$ is true." *Example*: If $P(x)$ is the property "x is a prime number," then $\{x|P(x)\} = \{x|\ x \text{ is a prime number}\}$ is the set of all prime numbers.

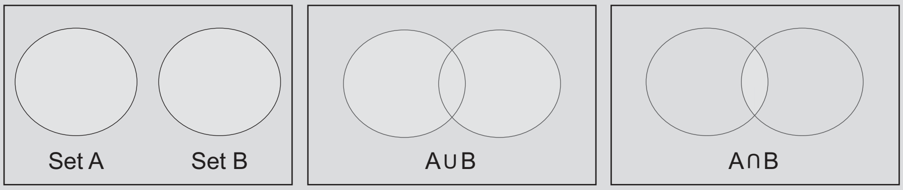
Set membership: Writing $x \in A$ indicates that x is a member of set A and writing $x \notin A$ means that it is not. These may be read "x is in A" and "x is not in A," respectively.

Equivalence: Sets are *equivalent* iff (which means "if and only if") they have precisely the same elements. We assume each element to be listed only once, so $\{a, a, b, c\}$ and $\{a, b, c\}$ are equivalent sets. Likewise, the order of elements does not matter, so $\{a, b, c\}$ and $\{b, a, c\}$ are equivalent sets.

Null set: The *null set* is the set having no elements, commonly denoted by $\emptyset$ or $\{\}$.

Subsets: Set B is a *subset* of set A if each element of B is in A. The notation $B \subset A$ indicates that B is a subset of A. Equivalently $A \supset B$ indicates that A is a superset of B (contains B as a subset). Every set has itself and the null set as subsets. If $B \subset A$ but A and B are not equivalent and $B \neq \{\emptyset\}$, B is a *proper subset* of A.

Union and intersection: The *union* $A \cup B$ of two sets A and B is the set containing all elements that are in A or in B, or in both A and B. The *intersection* $A \cap B$ of sets A and B is the set of only those elements contained in both A and B. The union and intersection of two sets are conveniently represented in terms of *Venn diagrams*.



Cartesian product: A cartesian product $X \times Y$ of sets X and Y is the set of ordered pairs (x, y) with $x \in X$ and $y \in Y$. The prototype is $\mathbb{R}^2 = \mathbb{R}^1 \times \mathbb{R}^1$, where $\mathbb{R}^1$ is the real number line and $\mathbb{R}^2$ is the 2D cartesian plane. If X is of dimension m and Y is of dimension n, the cartesian product $X \times Y$ is a set of dimension $m \times n$.

More advanced properties such as open and closed sets, or equivalence classes for sets, will be addressed as they are needed (for example, in Box 24.1).

Group "multiplication" is a law of binary combination that can be much more abstract than ordinary arithmetic multiplication. The group definition requires associativity but not commutivity. If a group has commutative elements so that $ab = ba$ for all a and b, the group is *abelian*; if it has non-commuting elements ($ab \neq ba$) the group is *non-abelian*.

2.3 Examples of Groups

The preceding requirements are relatively easy to satisfy and many sets can form groups under some law of binary combination. Let us give a few simple examples.

2.3.1 Additive Group of Integers

The set of all integers $G = \{\ldots, -3, -2, -1, 0, 1, 2, 3, \ldots\}$ forms a group under the binary operation of ordinary arithmetic addition.

- The sum of any two integers is an integer.
- A unique identity element $e = 0$ exists.
- A unique inverse exists for each element since $-n + n = e = 0$.
- Arithmetic addition is associative.

This is called the *additive group of integers* Z; it is abelian since addition commutes. Whether a set forms a group under some binary operation depends both on the set members and the binary operation.

Example 2.3 The integers are a group under addition but the positive integers are not (elements have no inverse). The real numbers are a group under addition but not under multiplication (zero has no inverse).

Box 2.2 The Two-Element Group

The group with two elements may be defined by listing the elements $\{e, a\}$, where e is the identity, and specifying the results of the multiplication operation. One way to do this is to give the *multiplication table* for elements of the group.

The Multiplication Table

From the properties of the identity we must have $ee = e$ and $ea = ae = a$ (for compactness the multiplication symbol often will be omitted: $ae \equiv a \cdot e$). Thus, only the product aa is required to complete the multiplication table. Either $aa = a$, or $aa = e$, if the closure property is to be satisfied. The first is impossible because multiplying the equation from both sides by a^{-1} gives $a = e$, which must be false if this is a two-element group.[a] So a is its own inverse, $aa = e$, and the *multiplication table* is

C_2	e	a
e	e	a
a	a	e

As indicated, this group is commonly denoted C_2, the *cyclic group* of order 2. Equivalently, it is the group Z_2 of integers 0 and 1 under addition modulo 2 (Problem 2.31).

Important Lessons

Two important properties of groups have been illustrated by this simple example.

1. Only *one* abstract group with two elements exists, since any two-element group must have the multiplication table just constructed.
2. Finite groups may be specified by their multiplication table, since this tells us both the elements of the set and the nature of the multiplication operation.

As you are asked to show in Problem 2.4, the group with three elements is also unique because the multiplication table is determined by general group properties independent of the specific group elements or nature of the binary combination law. The two-element and three-element groups illustrate the abstract nature of groups, since their properties are fixed by the group postulates, independent of specifics.

[a] This proof can be generalized to the useful rule that in a group multiplication table each group element must appear exactly once in each row and exactly once in each column of multiplication results. This may be viewed as a special case of the *rearrangement lemma* discussed in Section 6.2.5.

The additive group of integers has an infinity of discrete elements. There also are groups where the elements form an uncountable continuum (*continuous groups*), and groups with a finite number of elements (*finite groups*). An example of a finite group is given in Box 2.2.

2.3.2 Rotation and Translation Groups

As another example of a group, consider rotations. A natural multiplication may be defined by taking the product of two operations to be the resultant of applying first one operation and then the other to an arbitrary physical system. The product of two transformations A

and B may be denoted by $AB \equiv A \cdot B$, with the convention that the operator B is to be applied first and then the operator A. For rotations the following is clear physically.

1. Two successive rotations on a physical system are equivalent to some other rotation.
2. Rotation of a system through the null angle corresponds to an identity operation.
3. A system may be rotated through an angle and then rotated back to the original orientation. The second operation defines the unique inverse of the first.
4. If three successive rotations are performed, the same result is obtained if we perform the first and then the resultant of the other two, or if we perform the resultant of the first two and then the third rotation. Thus, rotation is associative.
5. For rotations in three dimensions, the result of two successive rotations about different axes is not necessarily equivalent to the result obtained by performing these rotations in reverse order. Thus, 3D rotations do not generally commute.

Therefore, 3D rotations form a *rotation group* that is non-abelian. This is a *continuous group* since the possible angles for a rotational transformation are continuous. Such groups of continuous transformations will be among the most important that we will examine. As another example of a continuous group, it is easily verified that the set of translations form a group under the natural definition that multiplication of two translations corresponds to applying first one and then the other translation operation to a physical system. The group of rotations and the group of translations are members of a special class of continuous groups called *Lie groups* that will occupy a prominent place in our discussion.

Sets of rotations and sets of translations each define continuous groups sharing many features, but they differ in two important aspects. The first is that translation groups are abelian but the 3D rotation group is non-abelian. The second concerns the group parameter space. Rotational parameters are real angles and the space is *bounded* (rotation by 2π about an axis returns us to the starting point). However, the parameter space for translations is *unbounded.* A continuous group for which the parameter space is closed and bounded is called *compact*; otherwise it is termed *non-compact*. As will be seen, the mathematical structure and the associated physical implications of compact and non-compact groups differ considerably; accordingly they have different roles to play in physical applications.

2.3.3 Parameterization of Continuous Groups

The number of elements is the *order* of a group, which can be finite or infinite. For finite groups of low order the group elements may simply be listed. This is inappropriate for continuous groups such as the rotation group, for which it is customary to specify a set of continuously varying parameters that describe the transformation: $\boldsymbol{\alpha} = (\alpha_1, \alpha_2, \ldots, \alpha_n)$. The continuous group associated with this parameter set is called an *n-parameter group*. The 3D rotations are a three-parameter group since there are three axes of rotation, each described by an independent angle. *Do not confuse the order of a group and the number of continuous parameters that define it.* For continuous groups the order is infinite but the number of parameters is usually finite. Our standard notation for continuous groups will be $G(\alpha_1, \alpha_2, \ldots, \alpha_n)$, where the n parameters α_n all vary continuously over some range.