Conformal Blocks, Generalized Theta Functions and the Verlinde Formula

In 1988, E. Verlinde gave a remarkable conjectural formula for the dimension of conformal blocks over a smooth curve arising from representations of affine Lie algebras. Verlinde's formula arose from physical considerations, but it attracted further attention from mathematicians when it was realized that the space of conformal blocks admits an interpretation as the space of generalized theta functions. A proof followed through the work of many mathematicians in the 1990s.

This book gives an authoritative treatment of all aspects of this theory. It presents a complete proof of the Verlinde formula and full details of the connection with generalized theta functions, including the construction of the relevant moduli spaces and stacks of G-bundles. Featuring numerous exercises of varying difficulty, guides to the wider literature and short appendices on essential concepts, it will be of interest to senior graduate students and researchers in geometry, representation theory and theoretical physics.

SHRAWAN KUMAR is John R. and Louise S. Parker Distinguished Professor of Mathematics at the University of North Carolina, Chapel Hill. He was an invited speaker at the 2010 International Congress of Mathematicians and was elected a Fellow of the American Mathematical Society in 2012. This is his third book.

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SHRAWAN KUMAR
University of North Carolina, Chapel Hill

CAMBRIDGE UNIVERSITY PRESS

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University Printing House, Cambridge CB2 8BS, United Kingdom

One Liberty Plaza, 20th Floor, New York, NY 10006, USA

477 Williamstown Road, Port Melbourne, VIC 3207, Australia

314–321, 3rd Floor, Plot 3, Splendor Forum, Jasola District Centre,
New Delhi – 110025, India

103 Penang Road, #05–06/07, Visioncrest Commercial, Singapore 238467

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www.cambridge.org
Information on this title: www.cambridge.org/9781316518168
DOI: 10.1017/9781108997003

First published 2022

A catalogue record for this publication is available from the British Library.

Library of Congress Cataloging-in-Publication Data
Names: Kumar, S. (Shrawan), 1953– author.
Title: Conformal blocks, generalized theta functions and
the Verlinde formula / Shrawan Kumar.
Description: Cambridge ; New York, NY : Cambridge University Press, 2022. |
Series: New mathematical monographs | Includes bibliographical references and index.
Identifiers: LCCN 2021029703 (print) | LCCN 2021029704 (ebook) |
ISBN 9781316518168 (hardback) | ISBN 9781108997003 (epub)
Subjects: LCSH: Lie algebras. | Affine algebraic groups. | Conformal invariants. |
Functions, Theta. | Fiber bundles (Mathematics) | Moduli theory. |
BISAC: MATHEMATICS / Topology
Classification: LCC QC20.7.L54 K86 2022 (print) | LCC QC20.7.L54 (ebook) |
DDC 512/.482–dc23
LC record available at https://lccn.loc.gov/2021029703
LC ebook record available at https://lccn.loc.gov/2021029704

ISBN 978-1-316-51816-8 Hardback

Cambridge University Press
978-1-316-51816-8 — Conformal Blocks, Generalized Theta Functions and the Verlinde Formula
Shrawan Kumar
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Preface

The main aim of this book is to give a self-contained proof of the Verlinde formula for the dimension of the space of conformal blocks and prove the connection between conformal blocks and generalized theta functions.

Let Σ be a smooth projective irreducible s-pointed ($s \geq 1$) curve of any genus $g \geq 0$ with marked points $\vec{p} = (p_1, \ldots, p_s)$ and let G be a simply-connected simple algebraic group with Lie algebra $\mathfrak{g}$. We fix a positive integer c called the *level* and let D_c be the set of dominant integral weights of $\mathfrak{g}$ of level at most c. We attach weights $\vec{\lambda} = (\lambda_1, \ldots, \lambda_s)$ (with each $\lambda_i \in D_c$) to the marked points $\vec{p}$, respectively. Associated to the triple $(\Sigma, \vec{p}, \vec{\lambda})$, there is the space $\mathscr{V}_\Sigma^\dagger(\vec{p}, \vec{\lambda})$ of *conformal blocks* (also called *space of vacua*), which is a finite-dimensional space given as the dual of $\mathfrak{g} \otimes \mathbb{C}[\Sigma \setminus \vec{p}]$-coinvariants of a tensor product of s copies of integrable highest-weight modules of level c with highest weights $\vec{\lambda}$ of the affine Kac–Moody Lie algebra $\hat{\mathfrak{g}}$ associated to $\mathfrak{g}$. This space is a basic object in rational conformal field theory arising from the Wess–Zumino–Witten model associated to G. Now, E. Verlinde gave a remarkable conjectural formula for the dimension of $\mathscr{V}_\Sigma^\dagger(\vec{p}, \vec{\lambda})$ in 1988. This conjecture was 'essentially' proved by a pioneering work of Tsuchiya–Ueno–Yamada, wherein they proved the *Factorization Theorem* and the *invariance of dimension* of the space of conformal blocks under deformations of the curve Σ, which allow one to calculate the dimension of the space of conformal blocks for a genus g curve from that of a genus $g-1$ curve. Thus, the problem gets reduced to a calculation on a genus 0 curve, i.e., on $\Sigma = \mathbb{P}^1$. The corresponding algebra for $\Sigma = \mathbb{P}^1$ is encoded in the fusion algebra associated to $\mathfrak{g}$ at level c, which gives rise to a proof of an explicit Verlinde dimension formula for the space $\mathscr{V}_\Sigma^\dagger(\vec{p}, \vec{\lambda})$.

Classical theta functions can be interpreted in geometric terms as global holomorphic sections of a certain determinant line bundle on the moduli space $\text{Pic}^{g-1}(\Sigma)$ of line bundles of degree $g-1$ on Σ. This has a natural non-abelian generalization, where one replaces the line bundles on Σ by principal G-bundles on Σ to obtain the parabolic moduli space $M^G_{\mathrm{par},\vec{\tau}}(\Sigma)$ (or stack $\mathbf{Parbun}_G\left(\Sigma,\vec{P}\right)$) and certain determinant line bundles over them. Holomorphic sections of these determinant line bundles over these moduli spaces or stacks are called the generalized theta functions (generalizing the classical theta functions).

The Verlinde dimension formula attracted considerable further attention from mathematicians and physicists when it was realized that the space of conformal blocks admits an interpretation as the space of generalized theta functions. This interpretation was rigorously established independently in the 'non-parabolic' case by Beauville–Laszlo (for the special case $G = \mathrm{SL}_n$), Faltings and Kumar–Narasimhan–Ramanathan (for general G); and in the 'parabolic' case by Pauly (for the special case $G = \mathrm{SL}_n$), Laszlo–Sorger (for classical G and G_2 for the stack) and here in this book it is proved for general G.

The theory has undergone tremendous development in various directions and connections with diverse areas abound. The Verlinde formula and the ideas behind its proof have found numerous applications, e.g., in the theory of moduli spaces of vector bundles (and, more generally, principal G-bundles) on curves, the multiplicative eigenvalue problem, rank-level duality, moduli of curves (just to name a few). The works leading to the Verlinde dimension formula and connection between conformal blocks and generalized theta functions, as well as various applications, are scattered through the literature. For example, apart from the research papers, there is a Bourbaki talk and also lecture notes by C. Sorger, and a monograph by Ueno. But there is no single source containing various developments in and around the Verlinde formula explaining both of its aspects: the space of conformal blocks and the space of generalized theta functions, with details of proofs. This book attempts to fill this void in the literature.

As mentioned above, we give a self-contained proof of the Verlinde formula for the dimension of the space of conformal blocks (derived from the Factorization Theorem and the invariance of dimension of the space of conformal blocks under deformations of Σ, among others) and full details of the connection between conformal blocks and generalized theta functions. The proofs require techniques from algebraic geometry, geometric invariant theory, representation theory of affine Kac–Moody Lie algebras, topology and Lie algebra cohomology.

The main results covered in this book are: propagation of vacua; Factorization Theorem; flat projective connection on the sheaf of conformal blocks (thereby its local freeness); explicit Verlinde dimension formula for the space of conformal blocks; uniformization theorem for the moduli stack of quasi-parabolic G-bundles; identification of parabolic G-bundles over Σ with equivariant bundles on a certain Galois cover $\hat{\Sigma}$ of Σ; Harder–Narasimhan reduction of G-bundles; Narasimhan–Seshadri theorem for topological realization of polystable G-bundles over Σ; construction of the moduli *space* of parabolic semistable G-bundles over Σ; canonical identification of the space of conformal blocks with the space of generalized theta functions (over both parabolic moduli space and moduli stack); an explicit determination of the Picard group of the moduli space (as well as moduli stack) of G-bundles; and higher cohomology vanishing of the determinant line bundles on the moduli space. In addition, Chapter 1 is devoted to recalling the basic theory of affine Kac–Moody Lie algebras and their representations; and construction of the associated groups and their flag varieties to the extent we need them in the book. We have also added four appendices: one on the Dynkin index, which plays an important role in the identification of determinant line bundles on the moduli space; and the second and the third giving a crash (and hopefully quite palatable) course on $\mathbb{C}$-space (and $\mathbb{C}$-group) functors and stacks. The fourth appendix (due to S. Mukhopadhyay) gives a survey of rank-level duality.

This book should be useful for senior graduate students, postdocs and faculty members interested in the interaction between algebraic geometry, representation theory, topology and mathematical physics. Depending upon the interests of the audience, parts of the book are suitable for a one-year advanced graduate course. We have added numerous exercises of varying difficulty at the end of practically each section. We do require some knowledge of representation theory of (finite-dimensional) semisimple Lie algebras (roughly Chapters II and VI of Humphreys (1972)) and some algebraic geometry (roughly the first three chapters of Hartshorne (1977)).

I am indebted to M. S. Narasimhan, who introduced me to this beautiful garden. I am also grateful to my collaborators on the subject: A. Boysal, M. S. Narasimhan, and A. Ramanathan, and to P. Belkale and V. Balaji for numerous consultations. My special thanks are also due to A. Boysal, B. Conrad, C. Damiolini, N. Nitsure, S. Mukhopadhyay and X. Zhu, who carefully looked at parts of the book and pointed out some errors and made various comments to improve the exposition. I gave a semester-long course covering Chapters 1 through 4 at the University of Sydney during Fall 2015. It is my pleasure to thank the audience, especially Anthony Henderson, Gustav Lehrer, Alex Molev, Hoel Queffelec, Oded Yacobi and Ruibin Zhang, for their comments. I also gave a series of lectures (covering parts of the book) at Duke University (2006–07); the University of Georgia, Athens (May 2010); the Université Claude Bernard Lyon 1 (June 2017); and the Tata Institute of Fundamental Research, Mumbai (January 2018). The feedback from the audiences in these institutions was helpful. Finally, I acknowledge the continued support from NSF over several years during which the book was written. The typing of the book from my handwritten manuscript was done by M. P. Raghavendra Prasad from Sriranga Digital Technologies, Srirangapatna. I also thank Neeraj Kumar for his help in some formatting issues.

I dedicate this book to my wife Shyama and our children, Neeraj and Niketa.