Cambridge University Press
978-1-316-51816-8 — Conformal Blocks, Generalized Theta Functions and the Verlinde Formula
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Excerpt
More Information

Introduction

There follows a more detailed description of the contents of the book. For simplicity and uniformity in this Introduction, we let G be a connected simply-connected simple algebraic group over $\mathbb{C}$ with Lie algebra $\mathfrak{g}$ (though many of the results in the text are proved, more generally, for connected reductive groups) and let $\hat{\Sigma}$ be a connected smooth projective curve with faithful action of a finite group A and we set $\Sigma := \hat{\Sigma}/A$.

Chapter 1. *Section 1.1* lays out the basic notation to be used throughout the book. It also includes the Yoneda Lemma.

Section 1.2 introduces the basic theory of affine Kac–Moody Lie algebras $\hat{\mathfrak{g}}$. The main result here is the classification of integrable highest-weight $\hat{\mathfrak{g}}$-modules.

In *Section 1.3* we realize the loop group $G((t))$, its various subgroups and the infinite Grassmannian as ind-schemes. Specifically, consider the functors which assign, to any $\mathbb{C}$-algebra R, the groups (or set)

$$G(R((t))), G(R[[t]]), G(R[t^{-1}]), G(R((t)))/G(R[[t]]).$$

Then we show that they are representable functors represented respectively by ind-schemes $\bar{G}((t)), \bar{G}[[t]], \bar{G}[t^{-1}], \bar{X}_G$. In fact, we show that all these ind-schemes are reduced. The construction of $\bar{X}_G$ for $G = \mathrm{SL}_N$ proceeds via the so-called special lattice functor.

In *Section 1.4* we construct and discuss the central extensions of the loop group. This is essentially obtained by exponentiating the integrable highest-weight $\hat{\mathfrak{g}}$-modules $\mathscr{H}(\lambda_c)$, thereby realizing these representations as projective representations of the loop group $\bar{G}((t))$. We further show that the $\mathbb{G}_m$-central extension $\hat{\bar{G}}_{\lambda_c}$ thus obtained splits over $\bar{G}[[t]]$ and $\bar{G}[t^{-1}]$. In fact, the splitting is unique as shown in Section 8.2.

Chapter 2. Let Σ be a reduced projective curve with at worst only nodal singularity and let $\vec{p} = (p_1, \ldots, p_s)$ be a collection of distinct marked smooth points of Σ. We fix a central charge $c > 0$ and associate integrable highest-weight $\hat{\mathfrak{g}}$-modules with highest weights $\vec{\lambda} = (\lambda_1, \ldots, \lambda_s)$ (all with central charge c) to the points $\vec{p}$, respectively. To this data, there is associated the *space of vacua* $\mathscr{V}^{\dagger}_{\Sigma}(\vec{p}, \vec{\lambda})$ defined as the space of certain invariants in the dual of the tensor product $\mathscr{H}(\vec{\lambda}) := \mathscr{H}(\lambda_1) \otimes \cdots \otimes \mathscr{H}(\lambda_s)$. This space of vacua is a fundamental object for this book. It is shown that it is a finite-dimensional space.

In *Section 2.2* we prove *propagation of vacua*, which asserts that adding one extra point to $\vec{p}$ and associating $\mathscr{H}(0)$ to this extra point does not change the space of vacua.

In *Section 2.3* we give a manifestly finite-dimensional expression for the space of vacua on $\Sigma = \mathbb{P}^1$ in terms of the action of sl_2 passing through the highest root space of $\mathfrak{g}$.

Chapter 3. In *Section 3.1* we prove the basic *Factorization Theorem*, which explicitly relates the space of vacua on an s-pointed curve $(\Sigma, \vec{p})$ of genus g with a single node with that of the space of vacua on the normalization $\tilde{\Sigma}$ (which is of genus $g - 1$) marked with $s + 2$ points.

In *Section 3.2* we recall the definition of the *Sugawara elements* in the completion of the enveloping algebra of the affine Kac–Moody Lie algebra $\tilde{\mathfrak{g}}$ (the non-completed version of $\hat{\mathfrak{g}}$). These elements allow us to give the action of the Virasoro algebra on any smooth representation of $\tilde{\mathfrak{g}}$.

In *Section 3.3* we sheafify the construction of the space of vacua for a family $\mathfrak{F}_T$ of s-pointed curves with formal parameters at the marked points parameterized by a smooth variety T. We show that this sheaf $\mathscr{V}_{\mathfrak{F}_T}(\vec{p}, \vec{\lambda})$ is a coherent sheaf of $\mathscr{O}_T$-modules. We also sheafify the Virasoro algebra to allow its action on the sheafified version $\mathscr{H}(\vec{\lambda})_T$ of the tensor product $\mathscr{H}(\vec{\lambda})$ of integrable highest-weight $\hat{\mathfrak{g}}$-modules.

Then, in *Section 3.4* we show that the sheaf $\mathscr{V}_{\mathfrak{F}_T}(\vec{p}, \vec{\lambda})$ for a smooth family is locally free and admits a functorial flat projective connection. This connection generalizes the Knizhnik–Zamolodchikov connection for $\Sigma = \mathbb{P}^1$.

In *Section 3.5*, using the stack of stable s-pointed connected curves of fixed genus g and the local freeness of $\mathscr{V}_{\mathfrak{F}_T}(\vec{p}, \vec{\lambda})$ for a smooth family (proved in the previous section), we show that the dimension of the space of vacua is independent of the choice of the marked points $\vec{p}$ as well as the connected smooth curve Σ, as long as the genus of Σ is fixed and of course $\vec{\lambda}$ is fixed. Let us denote this dimension by $F_g(\vec{\lambda})$. Further, using the 'smoothing deformation' and the Factorization Theorem, we extend the above result to allow curves with nodes. This enables us to prove the following inductive formula to calculate the dimension $F_g(\vec{\lambda})$:

$$F_g(\vec{\lambda}) = \sum_{\mu} F_{g-1}(\vec{\lambda}, \mu^*, \mu),$$

where μ runs over dominant integral weights of $\hat{\mathfrak{g}}$ with central charge c. Thus, the problem to calculate $F_g(\vec{\lambda})$ for any g reduces to that for $g = 0$, i.e., $\Sigma = \mathbb{P}^1$ (though with $s + 2g$ marked points). Using a similar decomposition, the problem of calculating $F_0(\vec{\lambda})$ with n marked points reduces to that for three marked points on $\mathbb{P}^1$.

Chapter 4. As mentioned above, to determine $F_g(\vec{\lambda})$ we only need to determine $F_0(\vec{\mu})$ with three marked points. To be able to calculate $F_0(\vec{\mu})$, a general algebraic framework in the form of *fusion ring* $\mathbb{Z}[A]$ is introduced in *Section 4.1*. It is shown that the corresponding complexified algebra $\mathbb{C}[A]$ is a (finite-dimensional) reduced algebra.

In *Section 4.2* we consider a specific fusion ring $R_c(\mathfrak{g}) := \mathbb{Z}[D_c]$, called the *fusion ring of* $\mathfrak{g}$ *at level* c, with product structure constants coming from F_0 with three marked points. Simple algebraic manipulation in this ring allows us to give an explicit expression for $F_g(\vec{\lambda})$ once we are able to explicitly determine the set of characters S_{D_c} of $\mathbb{C}[D_c]$ (i.e., algebra homomorphisms to $\mathbb{C}$). This section is devoted to solve this problem by using the combinatorics of the affine Weyl group and its action on the dual $\mathfrak{h}^*$ of the Cartan subalgebra $\mathfrak{h}$ of $\mathfrak{g}$. One other important ingredient in determining S_{D_c} is the result that a certain linear map ξ_c from the representation ring $R(\mathfrak{g})$ of $\mathfrak{g}$ to the fusion ring $R_c(\mathfrak{g})$ at level c is a ring homomorphism. To prove that ξ_c is a ring homomorphism, we use the affine analogue of the Borel–Weil–Bott (BWB) theorem as well as a Lie algebra cohomology vanishing result of Teleman. (For the classical $\mathfrak{g}$ as well as $\mathfrak{g}$ of type G_2, there is a more direct proof that ξ_c is a ring homomorphism avoiding the Lie algebra cohomology vanishing result, as shown in Exercises 4.2.E.) Once we have explicitly determined S_{D_c} (as we have), one of the most important results of the book – the *Verlinde dimension formula* – follows easily by using simple representation theory for finite groups.

Chapter 5. Let $\mathfrak{S}$ be the category of quasi-compact separated schemes over $\mathbb{C}$ and let $\mathbf{Bun}_G(\Sigma)$ be the groupoid fibration over $\mathfrak{S}$ whose objects over $S \in \mathfrak{S}$ are G-bundles on $\Sigma \times S$ and morphisms are the G-bundle morphisms. Similarly, for an s-pointed curve $(\Sigma, \vec{p})$ together with a choice of standard parabolic subgroups $\vec{P} = (P_1, \ldots, P_s)$ attached to the marked points, we define the groupoid fibration $\mathbf{Parbun}_G(\Sigma, \vec{P})$ of quasi-parabolic G-bundles of type $\vec{P}$ over $(\Sigma, \vec{p})$. Then, as stated in *Section 5.1*, both these are smooth (algebraic) stacks.

In *Section 5.2* we define a 'tautological' G-bundle $\mathbf{U}$ over $\Sigma \times \bar{X}_G$, where (as earlier) $\bar{X}_G$ is the infinite Grassmannian. Consider the functor which assigns to any $\mathbb{C}$-algebra R the group $\text{Mor}(\Sigma^* \times \text{Spec } R, G)$, where $\Sigma^* := \Sigma \backslash \vec{p}$. Then we show that it is a representable functor represented by an ind-affine group variety denoted $\bar{\Gamma} = \bar{\Gamma}_{\vec{p}}$. Then we prove the *Uniformization Theorem* for both $\mathbf{Bun}_G(\Sigma)$ and $\mathbf{Parbun}_G(\Sigma, \vec{P})$. Specifically, they are realized as quotient stacks:

$$\mathbf{Bun}_G(\Sigma) \simeq \left[\bar{\Gamma} \backslash \bar{X}_G\right]$$

and

$$\mathbf{Parbun}_G(\Sigma, \vec{P}) \simeq \left[\bar{\Gamma} \backslash \left(\bar{X}_G \times \Pi_{i=1}^s (G/P_i)\right)\right],$$

where $\bar{\Gamma} = \bar{\Gamma}_\infty$ for a single point $\infty \in \Sigma$ different from any p_i in $\vec{p}$. An important ingredient in the proof of the above two uniformization theorems is a result due to Drinfeld–Simpson asserting that for a family of G-bundles over Σ parameterized by a scheme S, the pull-back of the family to some étale cover $\tilde{S}$ of S is trivial restricted to any affine open subset of Σ.

As an immediate consequence of the uniformization theorems specialized to $\text{Spec}\, \mathbb{C}$, we get the following bijections:

$$\text{Bun}_G(\Sigma) \simeq \Gamma \backslash X_G$$

and

$$\text{Parbun}_G(\Sigma, \vec{P}) \simeq \Gamma \backslash \left(X_G \times \Pi_{i=1}^s (G/P_i)\right),$$

where Bun_G (resp. Parbun_G) denotes the set of isomorphism classes of G-bundles (resp. quasi-parabolic G-bundles) over Σ and $\Gamma := \bar{\Gamma}(\mathbb{C})$ and similarly $X_G := \bar{X}_G(\mathbb{C})$.

Chapter 6. In *Section 6.1* we define the *stability, semistability* and *polystability* of vector bundles over Σ and extend these notions to G-bundles over Σ. The semistabilty of a G-bundle is equivalent to the semistability of its adjoint bundle. More generally, we define the *parabolic stability* and *parabolic semistability* for parabolic G-bundles over an s-pointed curve $(\Sigma, \vec{p})$. We extend the notions of stability, semistability and polystability to *A-stability, A-semistability* and *A-polystability* in the case a finite group A acts faithfully on a smooth projective curve $\hat{\Sigma}$. The main aim of this section is to prove an equivalence between the groupoid fibration of A-equivariant G-bundles on $\hat{\Sigma}$ and quasi-parabolic G-bundles on $(\Sigma, \vec{p})$, where $\vec{p} \subset \Sigma$ denotes the set of all the ramification points under the action of A on $\hat{\Sigma}$. But first we need to define the *local type* of A-equivariant G-bundles, which is achieved by proving the following result.

Let A act on the formal disc $\mathbb{D} := \operatorname{Spec}(\mathbb{C}[[t]])$. For a $\mathbb{C}$-algebra R, let $\mathbb{D}_R := \operatorname{Spec}(R[[t]])$ be the formal disc over $\operatorname{Spec} R$ and let $\mathscr{E}$ be an A-equivariant G-bundle over $\mathbb{D}_R$ (with the trivial action of A on R) such that it is trivial as a G-bundle. Then, there is a trivialization of the G-bundle $\mathscr{E}$ such that the action of A is a product action, i.e.,

$$\gamma \odot (x, g) = (\gamma x, \theta_\gamma(x(0))g), \text{ for } \gamma \in A, x \in \mathbb{D}_R \text{ and } g \in G,$$

for a morphism $\theta_\gamma \colon \operatorname{Spec} R \to G$. For any $x^o \in \operatorname{Spec} R$, we get a group homomorphism $\theta(x^o) \colon A \to G$ taking $a \mapsto \theta_\gamma(x^o)$. This homomorphism $\theta(x^o)$ is unique up to conjugation.

Now, given an A-equivariant G-bundle E over $\hat{\Sigma}$, for any ramification point $p_i \in \Sigma$, we take a point $\hat{p}_i$ in $\hat{\Sigma}$ over p_i. Applying the above result to the restriction of E to the formal disc in $\hat{\Sigma}$ around $\hat{p}_i$ and A replaced by the isotropy subgroup A_{p_i} of A at $\hat{p}_i$ (which is a cyclic group and, up to a conjugation, does not depend upon the choice of $\hat{p}_i$ over p_i), we get a homomorphism $A_{p_i} \to G$ (unique up to conjugation). This is, by definition, the *local type of E at p_i*. Let $\vec{p} = (p_1, \ldots, p_s)$ be the set of all the ramification points in Σ and let $\vec{\tau} := (\tau_1, \ldots, \tau_s)$ be the local type respectively at $\vec{p}$. Similar to the definition of the stack $\mathbf{Bun}_G(\Sigma)$, define the groupoid fibration $\mathbf{Bun}_G^{A,\vec{\tau}}(\hat{\Sigma})$ of A-equivariant G-bundles over $\hat{\Sigma}$ of local type $\vec{\tau}$, whose objects over any scheme S are A-equivariant G-bundles E_S over $\hat{\Sigma} \times S$ (A acting trivially on S) such that for any $t \in S$, $E_{S|\hat{\Sigma}\times t}$ is of local type $\vec{\tau}$. Then we prove that $\mathbf{Bun}_G^{A,\vec{\tau}}(\hat{\Sigma})$ is a stack and there is an isomorphism of stacks:

$$\mathbf{Bun}_G^{A,\vec{\tau}}(\hat{\Sigma}) \simeq \mathbf{Parbun}_G(\Sigma, \vec{P}),$$

where $\vec{P}$ corresponds to the Kempf parabolic subgroups attached to $\vec{\tau}$. In particular, $\mathbf{Bun}_G^{A,\vec{\tau}}(\hat{\Sigma})$ is also a smooth (algebraic) stack. Moreover, specializing the above isomorphism of stacks to $\operatorname{Spec}\mathbb{C}$, we get a bijective correspondence between the set of isomorphism classes of A-equivariant G-bundles on $\hat{\Sigma}$ of local type $\vec{\tau}$ with the set of isomorphism classes of quasi-parabolic G-bundles over Σ of type $\vec{P}$:

$$\operatorname{Bun}_G^{A,\vec{\tau}}(\hat{\Sigma}) \simeq \operatorname{Parbun}_G(\Sigma, \vec{P}).$$

Under this correspondence, A-semistable (resp. A-stable) G-bundles over $\hat{\Sigma}$ correspond to the parabolic semistable (resp. stable) bundles over Σ with the parabolic weights given by $\vec{\tau}$.

In *Section 6.2* we discuss *Harder–Narasimhan (HR) reduction* (also called *canonical reduction*) of any G-bundle E over Σ. A P-subbundle E_P of E (for a standard parabolic subgroup P of G) is called a HN reduction if the associated $L \simeq P/U$-bundle $E_P(P/U)$ is a semistable L-bundle and for any nontrivial character λ of P which lies in the positive cone generated by the simple roots of $\mathfrak{g}$,

$$\deg(E_P \times^P \mathbb{C}_\lambda) > 0,$$

where L is a Levi subgroup of P and U is the unipotent radical of P. We prove that for any G-bundle E over Σ, the HN reduction E_P exists and is unique. As a consequence, it is shown that for an embedding $G \hookrightarrow G'$ of connected reductive groups and a G-bundle E over Σ, if $E(G')$ is semistable then so is E. Conversely, if E is semistable then so is $E(G')$ if G is not contained in any proper parabolic subgroup of G'. As another consequence (cf. Exercises 6.2.E), one gets that an A-equivariant G-bundle over $\hat{\Sigma}$ is A-semistable if and only if it is semistable.

Section 6.3 is devoted to constructing stable (more generally, polystable) G-bundles over Σ topologically from a homomorphism $\rho\colon \pi_1(\Sigma) \to K \subset G$ of the fundamental group, where K is a maximal compact subgroup of G. Specifically, define the corresponding holomorphic G-bundle over Σ by

$$E_\rho := \tilde{\Sigma} \times^{\pi_1(\Sigma)} G \to \Sigma,$$

where $\tilde{\Sigma}$ is the simply-connected cover of Σ. These bundles E_ρ are called *unitary bundles*. Then it is shown that E_ρ is semistable. Further, for two such homomorphisms ρ and ρ', the bundles E_ρ and $E_{\rho'}$ are isomorphic if and only if ρ and ρ' are conjugate. Moreover, E_ρ is stable if and only if ρ is irreducible in the sense that the image of ρ is not contained in any proper parabolic subgroup of G. The irreducibility of ρ is also shown to be equivalent to the corresponding adjoint representation $\operatorname{ad}\rho$ having no $\pi_1(\Sigma)$-invariants (assuming G to be semisimple). The proof requires, in particular, an identification of a certain group cohomology of $\pi_1(\Sigma)$ with the cohomology of a certain vector bundle over Σ. Because of the standard presentation of $\pi_1(\Sigma)$, the set of all homomorphisms from $\pi_1(\Sigma) \to K$ can be identified with $\beta^{-1}(e)$, where

$$\beta\colon K^{2g} \to K,\ \big((h_1,k_1),\ldots,(h_g,k_g)\big) \mapsto \Pi_{i=1}^{g}[h_i,k_i],$$

where g is the genus of Σ. For $\bar{\rho} \in \beta^{-1}(e)$, let ρ be the corresponding representation of $\pi_1(\Sigma)$. Then it is shown that ρ is irreducible if and only if the tangent map $(d\beta)_{\bar{\rho}}$ is of maximal rank. In particular,

$$M_g(K) := \{\bar{\rho} \in \beta^{-1}(e) : \rho \text{ is irreducible}\}$$

is a smooth manifold of dimension $(2g - 1)\dim K$ (for semisimple G). Moreover, it supports an $\mathbb{R}$-analytic family of holomorphic G-bundles over Σ such that its Kodaira–Spencer infinitesimal deformation map is surjective everywhere. We now come to the following celebrated result (generalization of the classical Narasimhan–Seshadri result to G-bundles).

Any stable G-bundle E over Σ (for $g \geq 2$) is realized as E_ρ for an irreducible representation $\rho : \pi_1(\Sigma) \to K$ (and conversely). In fact, the result is valid for any connected reductive G provided we assume that E is of degree 0. The result can easily be extended for any polystable G-bundle E. Conversely, for any (not necessarily irreducible) representation $\rho : \pi_1(\Sigma) \to K$, E_ρ is polystable.

Let us point out the main strategy behind its proof. Let $\mathscr{F} \to \Sigma \times T$ be a $\mathbb{C}$-analytic family of stable G-bundles over Σ. Then we prove that

$$T_u := \{t \in T : \mathscr{F}_t \simeq E_\rho \text{ for some unitary representation } \rho\}$$

is a closed subset of T. Moreover, for any $\mathbb{R}$-analytic family of holomorphic G-bundles over Σ,

$$T_o := \{t \in T : \mathscr{F}_t \simeq E_\rho \text{ for some unitary irreducible } \rho\}$$

is an open subset of T, which follows from the surjectivity of the Kodaira–Spencer infinitesimal deformation map (mentioned above). Further, there exists an irreducible representation $\rho_o : \pi_1(\Sigma) \to K$ (this is where we need $g \geq 2$). Finally, we construct a $\mathbb{C}$-analytic family $\mathscr{E}$ of stable holomorphic G-bundles over Σ parameterized by a connected open subset V of $\mathbb{C}$ containing $\{0, 1\}$ such that $\mathscr{E}_0 \simeq E$ and $\mathscr{E}_1 \simeq E_{\rho_o}$. Observe that, for this family, $T_u = T_o$ since the family consists of stable bundles and (as observed above) ρ is irreducible if and only if E_ρ is stable. Now, V being connected and $T_u = T_o$ being both open and closed and nonempty, $T_o = V$. This proves that $E \simeq E_\rho$ for some irreducible ρ.

We extend these results to the setting of A-equivariant G-bundles over $\hat{\Sigma}$. Specifically, let π_1 be the fundamental group of $\hat{\Sigma}$. Then, π_1 is a normal subgroup of a group π such that π acts on the simply-connected cover $\tilde{\hat{\Sigma}}$ of $\hat{\Sigma}$ (having fixed points in general) with $\tilde{\hat{\Sigma}}/\pi \simeq \Sigma$ and $\pi/\pi_1 \simeq A$. Given a representation $\hat{\rho} : \pi \to K$, we can construct (as above) the holomorphic G-bundle over $\hat{\Sigma}$:

$$\hat{E}_{\hat{\rho}} := \tilde{\hat{\Sigma}} \times^{\pi_1} G \to \hat{\Sigma}.$$

Since $\hat{\rho}$ is a representation of π, $\hat{E}_{\hat{\rho}}$ acquires the canonical structure of an A-equivariant G-bundle. These bundles $\hat{E}_{\hat{\rho}}$ are called *A-unitary*. Conversely, if $\hat{E}_{\hat{\rho}}$ (the definition of which only requires the homomorphism $\hat{\rho}_{|\pi_1}$) acquires the structure of an A-equivariant G-bundle, then $\hat{\rho}_{|\pi_1}$ extends to π. We extend various results proved for E_ρ to that for $\hat{E}_{\hat{\rho}}$. In particular, $\hat{E}_{\hat{\rho}}$ is A-stable if and only if $\hat{\rho}$ is irreducible. Moreover, for any homomorphism $\hat{\rho}: \pi \to K$, $\hat{E}_{\hat{\rho}}$ is A-semistable (in fact, A-polystable). Conversely, we have the following equivariant generalization of the Narasimhan–Seshadri theorem for any G:

Any A-polystable G-bundle $\hat{E}$ over $\hat{\Sigma}$ (when the genus $g \geq 2$ of Σ) is realized as $\hat{E}_{\hat{\rho}}$ for a representation $\hat{\rho}: \pi \to K$ (and conversely).

Let $G \to \mathrm{GL}_V$ be a representation with finite kernel. Then we show that an A-equivariant G-bundle $\hat{E}$ over $\hat{\Sigma}$ is A-unitary if and only if the corresponding vector bundle $\hat{\Sigma}(V)$ is A-unitary.

Chapter 7. Let us first recall the following result due to Grothendieck.

Let X be a projective scheme with a very ample line bundle $\mathscr{L}$ over X and let E be a coherent sheaf on X. Then, for any fixed polynomial $P(z) \in \mathbb{Q}[z]$, define a contravariant functor which associates to any noetherian scheme S, set of all $\mathscr{O}_{X\times S}$-module quotients $\mathscr{F}$ of $E \boxtimes \mathscr{O}_S$ such that $\mathscr{F}$ is flat over S and $\mathscr{F}_{|X\times t}$ has Hilbert polynomial $P(z)$ (with respect to $\mathscr{L}$) for any $t \in S$. Then, this functor is representable by a projective scheme $Q = Q(E,P)$ called the *quot scheme*. Moreover, there is a 'tautological' coherent sheaf $\mathscr{U}$ over $X \times Q$.

Take a pair of positive integers (r,d) such that $d > r(2g-1)$, where g is the genus of Σ. We specialize the above general result to $X = \Sigma, E = \mathscr{O}_\Sigma \otimes \mathbb{C}^N$ and $P(z) = N + rhz$, where $N := d + r(1-g)$ and h is the degree of a fixed very ample line bundle over Σ. Thus, we get the quot scheme $Q = Q(E, P(z))$ and the tautological coherent sheaf $\mathscr{U}$ over $\Sigma \times Q$. Moreover, GL_N acts canonically on Q making $\mathscr{U}$ a GL_N-equivariant sheaf (with the trivial action of GL_N on Σ). Define the subset

$$R^{ss} := \{q \in Q : \bar{q} \text{ is a semistable vector bundle over } \Sigma \text{ and } \mathbb{C}^N = H^0(\Sigma, E) \to H^0(\Sigma, \bar{q}) \text{ is an isomorphism}\},$$

where $\bar{q}$ is the restriction $\mathscr{U}_{|\Sigma\times q}$. Then we prove that R^{ss} is a GL_N-stable irreducible smooth open subset of Q and $\mathscr{U}_{|\Sigma\times R^{ss}}$ is a rank-r vector bundle. Let $\mathscr{M}(r,d)$ (resp. $\mathscr{M}^s(r,d)$) be the functor of semistable (resp. stable) vector bundles over Σ of rank r and degree d. Then the main result of *Section 7.1* asserts that $\mathscr{M}(r,d)$ has a coarse moduli space $M(r,d) := R^{ss}//\mathrm{SL}_N$, which is an irreducible, normal projective variety with rational singularities of dimension $(g-1)r^2+1$ if $g \geq 2$. Moreover, the subfunctor $\mathscr{M}^s(r,d)$ has a coarse moduli space $M^s(r,d)$, which is an open subset of $M(r,d)$. The canonical map $\mathscr{M}(r,d)(\operatorname{Spec}\mathbb{C}) \to M(r,d)$ is surjective such that its fibers are precisely the equivalence classes of semistable vector bundles and its restriction $\mathscr{M}^s(r,d)(\operatorname{Spec}\mathbb{C}) \to M^s(r,d)$ is a bijection.

In *Section 7.2* we extend the above results for vector bundles to G-bundles and even more generally to A-equivariant G-bundles over $\hat{\Sigma}$. To this end, we fix an embedding $i: G \hookrightarrow \mathrm{SL}_r \subset \mathrm{GL}_r$ and realize G-bundles as GL_r-bundles via the embedding i (equivalently, rank-r vector bundles) together with a G-subbundle. To take into consideration the A-action, we fix an A-stable finite subset $\{y_1, \dots, y_b\}$ of $\hat{\Sigma}$ and a positive integer d' such that the divisor $\vec{y} := d' \sum_j y_j$ has degree $\geq 2\hat{g}$, $\hat{g}$ being the genus of $\hat{\Sigma}$. Now, we consider the quot scheme as above:

$$Q = Q\big(E = \big(\mathscr{O}_{\hat{\Sigma}} \otimes \mathbb{C}^N\big) \otimes \mathscr{O}_{\hat{\Sigma}}(-\vec{y}), P(z)\big),$$

together with the tautological sheaf $\mathscr{U}$ over $\hat{\Sigma} \times Q$, where $N := r(d+1-\hat{g})$ and $P(z) = r(1-\hat{g}) + rhz$ (h being the degree of a fixed very ample A-equivariant line bundle H over $\hat{\Sigma}$). Depending on the fixed local type $\vec{\tau}$ of A-equivariant G-bundles over $\hat{\Sigma}$, we fix a representation $\mathring{\tau}$ of A on $\mathbb{C}^N$. In fact, this representation of A on $\mathbb{C}^N$ is obtained from taking any A-semistable G-bundle F over $\hat{\Sigma}$ of topological type $\vec{\tau}$ and then taking the action of A on $\mathbb{C}^N \simeq H^0(\hat{\Sigma}, F(\vec{y}))$. (This action of A does not depend upon the choice of F.) This gives rise to a canonical action of A on Q making $\mathscr{U}$ an A-equivariant sheaf. Define the subset

$$\begin{aligned} R^{ss}_{\mathring{\tau}} := \{q \in Q^A : \bar{q} &\text{ is an } A\text{-semistable vector bundle over } \hat{\Sigma} \text{ and} \\ \mathbb{C}^N = H^0\big(\hat{\Sigma}, E(\vec{y})\big) &\to H^0\big(\hat{\Sigma}, \bar{q}(\vec{y})\big) \text{ is an isomorphism}\}, \end{aligned}$$

where Q^A is the subscheme of A-invariants in Q and $\bar{q}$ is the restriction $\mathscr{U}_{|\hat{\Sigma}\times q}$. Let $\mathscr{U}^{ss}_{\mathring{\tau}}$ denote the restriction $\mathscr{U}_{|\hat{\Sigma}\times R^{ss}_{\mathring{\tau}}}$. Let $\mathfrak{G} := \mathrm{GL}_N^A$, the A-invariants under the conjugation action of A on GL_N induced from the representation $\mathring{\tau}$. Then, $R^{ss}_{\mathring{\tau}}$ is a $\mathfrak{G}$-stable open subset of Q^A and $\mathscr{U}^{ss}_{\mathring{\tau}}$ is an A-equivariant rank-r vector bundle with the action of $\mathfrak{G}$.

For any scheme S, let $\mathfrak{S}_S$ be the category consisting of morphisms $T \to S$ as objects and S-morphisms between them as morphisms. Define the contravariant functor to the category of sets (abbreviating the frame bundle of $\mathscr{U}^{ss}_{\mathring{\tau}}$ by $\mathscr{F}$):

$$\Gamma(i, \mathscr{F}) \colon \mathfrak{S}_{R^{ss}_{\mathring{\tau}}} \to \textbf{Set}$$

by $\Gamma(i,\mathscr{F})(f : T \to R^{ss}_{\mathring{\tau}})$ = the set of A-equivariant sections σ of $\mathscr{F}_f/G$, where $\mathscr{F}_f := \big(\mathrm{Id}_{\hat{\Sigma}} \times f\big)^*(\mathscr{F})$. Clearly, giving any such section σ is equivalent to giving an A-equivariant G-subbundle $\mathscr{F}_f(\sigma)$ of $\mathscr{F}_f$ over $\hat{\Sigma} \times T$. For an A-equivariant topological G-bundle τ over $\hat{\Sigma}$, define a subfunctor

$\Gamma^\tau(i,\mathscr{F})\colon \mathfrak{S}_{R^{ss}_{\mathring{\tau}}} \to$ **Set** of $\Gamma(i,\mathscr{F})$ by demanding that for any $f\colon T \to R^{ss}_{\mathring{\tau}}$, the G-subbundle $\mathscr{F}_f(\sigma)$ restricted to any $t \in T$ is topologically A-equivariant isomorphic with τ. Then, by a general result, the functor $\Gamma^\tau(i,\mathscr{F})$ is representable by a separated scheme of finite type $f_\tau\colon R^{ss}_\tau(G) \to R^{ss}_{\mathring{\tau}}$. Moreover, there exists a 'universal' A-equivariant G-bundle

$$\mathscr{U}^{ss}_\tau(G) \in \Gamma^\tau(i,\mathscr{F})(f_\tau) \text{ over } \hat{\Sigma} \times R^{ss}_\tau(G).$$

Now, the main result of this section asserts that the A-semistable G-bundles over $\hat{\Sigma}$ of topological type τ admit a coarse moduli space

$$M^G_\tau(\hat{\Sigma}) := R^{ss}_\tau(G)//\mathfrak{G}.$$

Moreover, it is an irreducible, normal variety with rational singularity; and nonempty and projective if the genus g of Σ is at least 2. We further prove that any element in $M^G_\tau(\hat{\Sigma})$ contains a unique A-polystable representative. Because of the correspondence between A-equivariant G-bundles over $\hat{\Sigma}$ and the parabolic G-bundles over Σ (as in Section 6.1), these results readily translate to the results about the moduli space of parabolic semistable G-bundles.

In the case $A = (1)$ so that $\hat{\Sigma} = \Sigma$, $M^G_\tau(\Sigma)$ is the (non-parabolic) moduli space of semistable G-bundles over Σ.

Chapter 8. Recall the definition of the ind-affine group variety $\bar{\Gamma}$ from above (summary of Chapter 5). Then, in *Section 8.1*, we prove that it is irreducible. The proof relies on showing that (under the analytic topology) Γ^{an} is path-connected, where $\Gamma := \bar{\Gamma}(\mathbb{C})$. As a corollary of this, we show that the infinite Grassmannian $\bar{X}_G$ is an irreducible ind-projective variety.

In *Section 8.2* we prove that the $\mathbb{G}_m$-central extension $\hat{\bar{G}}_{\lambda_c}$ described above in the summary of Section 1.4 splits uniquely for $\lambda_c = 0_c$ over $\bar{\Gamma} = \bar{\Gamma}_p$ for a single point $p \in \Sigma$.

Section 8.3: we prove that the space of vacua $\mathscr{V}^\dagger_\Sigma(\vec{p},\vec{\lambda})$ for any s-pointed smooth curve $(\Sigma,\vec{p})$ is canonically identified (up to scalar multiples) with the space of global sections of the moduli stack $\mathbf{Parbun}_G(\Sigma,\vec{P})$ with respect to a certain line bundle $\mathscr{L}(\vec{\lambda})$, where the parabolic subgroups $\vec{P}$ and $\mathscr{L}(\vec{\lambda})$ are given in terms of $\vec{\lambda}$ and the central charge c. The main ingredient in the proof is the analogue of the Borel–Weil theorem for affine Lie algebras and the propagation of vacua. We also explicitly determine the Picard group of the moduli stack $\mathbf{Parbun}_G(\Sigma,\vec{P})$.

Section 8.4: we first define the determinant and theta line bundles ($\mathrm{Det}(\mathscr{V})$ and $\Theta(\mathscr{V})$, respectively) of a family $\mathscr{V}$ of vector bundles over Σ parameterized by a noetherian scheme S. These are line bundles over S. Thus, for a family