Introduction

This monograph presents the theory of global attractors and of the long-time behavior of solutions of nonlinear Hamiltonian partial differential equations in infinite space. This theory was initiated by one of the authors in 1990, and it has been developed in collaboration with H. Spohn since 1995 and with V. S. Buslaev, A. Comech, V. Imaikin, E. Kopylova, D. Stuart, and B. Vainberg since 2005. The theory resulted, in particular, in the first rigorous solution of the problem of radiation damping in classical electrodynamics and in the first rigorous model of Bohr's transitions between quantum stationary states. This progress became possible due to novel application of subtle methods of the Wiener Tauberian theorem and the Titchmarsh convolution theorem.

The theory of attractors for nonlinear PDEs began in Landau's famous 1944 paper [22], where he proposed the first mathematical interpretation of the onset of turbulence as the growth of the dimension of attractors of the Navier–Stokes equations when the Reynolds number increases.

The foundation for the corresponding mathematical theory was laid in 1951 by Hopf, who first established the existence of global solutions of the 3D Navier–Stokes equations [5]. He introduced the *method of compactness*, which is a nonlinear version of Faedo–Galerkin approximations. This method is based on a priori estimates and Sobolev embedding theorems and has had an essential influence on the development of the theory of nonlinear PDEs (see [2, 3, 12]).

The modern development of the theory of global attractors for *dissipative PDEs*, that is, PDEs with friction, originated in 1975–1985 in publications by J. Ball, C. Foias, J. M. Ghidaglia, J. K. Hale, D. Henry, and R. Temam and was developed further by M. I. Vishik, A. V. Babin, V. V. Chepyzhov, A. Haraux, A. A. Ilyin, A. Miranville, V. Pata, E. Titi, S. Zelik, and others. An essential part of the theory up to 2000 was covered in the monographs [16]–[23].

One of the central subjects of research in this theory is the global attractor of all bounded subsets of the corresponding Banach phase space. Typically, this attractor is a submanifold connecting stationary states, which is an analog of separatrices. Each single point also attracts to this submanifold and eventually converges to one of stationary states,

$$\psi(x,t) \to S(x), \qquad t \to +\infty, \tag{1}$$

where the convergence holds in appropriate norm on the Banach phase space. In particular, the *relaxation to an equilibrium regime* in chemical reactions is due to energy dissipation.

The results obtained concern a wide class of nonlinear *dissipative* PDEs, including fundamental equations of applied and mathematical physics: the Navier–Stokes equations, nonlinear parabolic equations, reaction–diffusion equations, wave equations with friction, integro-differential equations, equations with delay, equations with memory, and so on. The techniques of functional analysis of nonlinear PDEs were developed for the study of the structure of different types of attractors; their smoothness and their fractal and Hausdorff dimensions; and their dependence on parameters, on averaging, and so on.

The development of a similar theory for *Hamiltonian PDEs* seemed at first to be unmotivated and even impossible in view of energy conservation and time reversal for these equations. However, it turned out that such a theory is possible, and its development was inspired by the problem of mathematical interpretation of basic postulates of quantum theory. These relations to quantum theory are discussed in the final chapter (Chapter 8). More details can be found in [214].

Results obtained between 1990 and 2020 suggest that long-time global attraction to a finite-dimensional submanifold in the corresponding Hilbert phase space is, in fact, a typical feature for nonlinear Hamiltonian PDEs in infinite space. These results are presented in our monograph.

For Hamiltonian PDEs in infinite space, the theory of attractors differs significantly from the case of dissipative systems, where the global attraction to stationary states is caused by an energy dissipation that is due to friction. For Hamiltonian PDEs the friction and energy dissipation are absent, and the global attraction is caused by radiation that irreversibly carries energy to infinity. This peculiarity required novel tools for analysis of nonlinear Hamiltonian PDEs, which are presented in this monograph.

Let us note, however, that this theory is only at an initial stage of its development and cannot be compared with the theory of attractors of dissipative PDEs with regard to richness and diversity of results.

The modern development of the theory of nonlinear Hamiltonian PDEs dates back to K. Jörgens [7], who first established the existence of global solutions for nonlinear wave equations of the form

$$\ddot{\psi}(x,t) = \Delta\psi(x,t) + f(\psi(x,t)), \qquad x \in \mathbb{R}^n, \tag{2}$$

by developing the Hopf method of compactness. Subsequent studies of the well-posedness for nonlinear PDEs were presented by J.-L. Lions [12] and by T. Cazenave and A. Haraux [2, 3].

The first results on *long-time asymptotics* for *linear hyperbolic PDEs* in infinite space were established in the scattering theory by P. D. Lax, C. S. Morawetz, and R. S. Phillips for the wave equation in the exterior of a star-shaped obstacle [31]. This is the *local energy decay*: for any finite $R > 0$,

$$\int_{|x|<R} [|\dot{\psi}(x,t)|^2 + |\nabla\psi(x,t)|^2 + |\psi(x,t)|^2]dx \to 0, \qquad t \to \pm\infty. \tag{3}$$

This decay means that the energy escapes each bounded region for large times. For general linear hyperbolic PDEs and systems, similar local decay was established by B. R. Vainberg [37]. The extension of this decay to *nonlinear Hamiltonian PDEs* was established first by I. Segal, C. S. Morawetz, and W. Strauss [32]–[36]. In these papers the local energy decay (3) was proved for solutions of equations (2) with small initial data in the case of *defocusing nonlinearities* similar to

$$f(\psi) = -m^2\psi - \varkappa|\psi|^{p-1}\psi, \tag{4}$$

where $m^2 \geq 0$, $\varkappa > 0$, and $p > 1$. Moreover, in these articles the corresponding nonlinear wave operators and scattering operators are constructed. In [77, 78], W. Strauss established the completeness of the scattering for small solutions of more general equations.

For convenience, characteristic properties of all finite-energy solutions of an equation will be referred to as *global* to distinguish them from the corresponding *local* properties of the solutions with initial data sufficiently close to an attractor. Note that global attraction to a (proper) attractor is impossible for finite-dimensional Hamiltonian systems because of energy conservation. All the aforementioned results [32]–[36] on local energy decay (3) for nonlinear Hamiltonian PDEs mean that the corresponding *local attractor* of solutions with small initial states consists of only the zero point.

Theory of global attractors The first results on *global attractors* for nonlinear Hamiltonian PDEs were obtained by one of the present authors in 1991–1995 for 1D equations [40, 41, 42] and were extended to multidimensional equations in 1995–2020 in collaboration with A. Comech, V. S. Buslaev, E. Kopylova, H. Spohn, D. Stuart, B. R. Vainberg, and others. These results were obtained from an analysis of the irreversible energy radiation to infinity, which plays the role of dissipation. This progress was achieved by a novel application of subtle methods of harmonic analysis: the Wiener Tauberian theorem, the Titchmarsh convolution theorem, the new theory of multipliers in the space of quasimeasures, and other methods.

The questions of asymptotic stability required the use of the stationary scattering theory of Agmon, Jensen, and Kato [171, 183] and of the eigenfunction expansion for non-selfadjoint Hamiltonian operators [137, 138] based on M. G. Krein's theory of J-selfadjoint operators.

One of the key observations is that the results obtained so far indicate a certain dependence of long-time asymptotics of solutions on the symmetry group of the equation. For example, it may be the trivial group $G = \{e\}$, or the group of translations $G = \mathbb{R}^n$, or the unitary group $G = U(1)$, or the orthogonal group $SO(3)$. This observation suggests general conjecture for nonlinear Hamiltonian *autonomous* PDEs of type

$$\dot{\Psi}(t) = F(\Psi(t)), \qquad t \in \mathbb{R}, \tag{5}$$

with a Lie symmetry group G, which acts on the Hilbert or Banach phase space $\mathcal{E}$ of the equation via a representation T.

Conjecture A (On attractors) *For* **generic** *nonlinear Hamiltonian PDEs (5) with a Lie symmetry group G, any finite-energy solution admits the asymptotics*

$$\Psi(t) \sim e^{\hat{\lambda}_\pm t}\Psi_\pm, \qquad t \to \pm\infty \tag{6}$$

in the appropriate topology of the phase space $\mathcal{E}$.

Here $\hat{\lambda}_\pm = T'(e)\lambda_\pm$, where $\lambda_\pm$ belong to the corresponding Lie algebra $\mathfrak{g}$, while the $\Psi_\pm(x)$ are some *limiting amplitudes* depending on the trajectory $\Psi(x,t)$ considered. Both pairs $(\Psi_+, \hat{\lambda}_+)$ and $(\Psi_-, \hat{\lambda}_-)$ are solutions of the corresponding *nonlinear eigenvalue problem* (3.11.5); see more details in Section 3.11.

Let us specify the asymptotics (6) for the four symmetry groups mentioned above.

1. Equations with trivial symmetry group $G = \{e\}$ For such *generic* equations, the conjecture (6) means *global attraction to stationary states*

$$\psi(x,t) \to S_\pm(x), \qquad t \to \pm\infty, \tag{7}$$

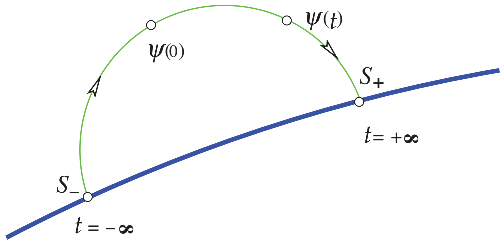

Figure 1 Convergence to stationary states.

as is illustrated in Figure 1. Here the states $S_\pm(x)$ depend on the trajectory $\psi(x,t)$ under consideration, and the convergence holds in local seminorms of type $L^2(|x| < R)$ with any $R > 0$. This convergence cannot hold in global norms (i.e., in norms corresponding to $R = \infty$) due to energy conservation. The asymptotics (7) can be symbolically written as the transitions

$$S_- \mapsto S_+, \tag{8}$$

which can be considered as the mathematical model of Bohr's quantum jumps (8.1.1).

Such an attraction was established in [40]–[52] for a variety of model equations: (1) for a string coupled to nonlinear oscillators, (2) for a 3D wave equation coupled to a charged particle and for the Maxwell–Lorentz equations, and also (3) for wave equations and Dirac and Klein–Gordon equations with concentrated nonlinearities.

All proofs rely on the bounds for radiation which irreversibly carries energy to infinity. The proofs of global attraction in [44, 45] rely on a novel application of the Wiener Tauberian theorem [15], which provides the relaxation of the acceleration of the particle

$$\ddot{q}(t) \to 0, \qquad t \to \pm\infty \tag{9}$$

under the *Wiener condition* (1.5.13) on the particle charge density. These results gave the first rigorous proof of *radiation damping* (9) in classical electrodynamics, which has been an open problem for about 100 years.

The results of [40]–[44] and [50] are presented with detail in Chapter 1.

In all problems considered here, the convergence (7) implies by the Fatou theorem the inequality

$$\mathcal{H}(S_\pm) \le \mathcal{H}(Y(t)) \equiv \text{const}, \quad t \in \mathbb{R}, \tag{10}$$

where $\mathcal{H}$ is the corresponding Hamiltonian (energy) functional. This inequality is an analog of the well-known property of the weak convergence in the Hilbert and Banach spaces. Simple examples show that strong inequality in (10) is possible, which means the irreversible scattering of energy to infinity.

Example 1 The d'Alembert waves In particular, the asymptotics (7) and the strong inequality (10) can easily be demonstrated for the d'Alembert equation with general solution $\psi(x,t) = f(x-t) + g(x+t)$. Namely, the local convergence $\psi(\cdot,t) \to 0$ in $L^2_{\rm loc}(\mathbb{R})$ obviously holds for all $f,g \in L^2(\mathbb{R})$. On the other hand, the convergence to zero in the global norm of $L^2(\mathbb{R})$ obviously fails if $f(x) \not\equiv 0$ or $g(x) \not\equiv 0$.

Example 2 Nonlinear strong Huygens principle Similarly, a solution of the 3D wave equation with unit speed of propagation is concentrated in spherical layers $|t|-R < |x| < |t|+R$ if the initial data have support in the ball $|x| \le R$. Therefore, the solution converges to zero in $L^2_{\rm loc}(\mathbb{R}^3)$ as $t \to \pm\infty$, although its energy remains constant. This also illustrates the strong inequality in (10). This convergence corresponds to the well-known *strong Huygens principle* in optics and acoustics (see [1]). Thus, global attraction to stationary states (7) is a generalization of the strong Huygens principle to nonlinear equations. The difference is that for the linear wave equation the limit is always zero, while for nonlinear equations the limit can be any stationary solution.

2. Equations with the symmetry group of translations $G = \mathbb{R}^n$ Let us consider, as an example, the case of the simplest representation

$$[T(a)\psi](x) := \psi(x-a), \qquad x \in \mathbb{R}^n \tag{11}$$

for $a \in \mathbb{R}^n$. Then the asymptotics (6) means *global attraction to solitons* (traveling waves)

$$\psi(x,t) \sim \psi_\pm(x - v_\pm t), \qquad t \to \pm\infty, \tag{12}$$

where the asymptotics holds in local seminorms of type $L^2(|x - v_\pm t| < R)$ with any $R > 0$, that is, *in the comoving frame of reference*.

Such soliton asymptotics was proved first for *integrable equations* (Korteweg–de Vries equation (KdV), etc.); see [53, 59]. Moreover, for the Korteweg–de Vries equation, more accurate soliton asymptotics in *global norms* with several solitons were first discovered by M. Kruskal and N. J. Zabuzhsky in 1965 by numerical simulation: it is the decay to solitons

$$\psi(x,t) \sim \sum_k \psi_\pm(x - v^k_\pm t) + w_\pm(x,t), \qquad t \to \pm\infty, \tag{13}$$

where $w_\pm$ are some dispersive waves. A trivial example is provided by the d'Alembert equation $\ddot\psi(x,t) = \psi''(x,t)$, for which any solution reads $\psi(x,t) = f(x-t) + g(x+t)$.

Later on, such asymptotics were proved by the method of the *inverse scattering problem* for nonlinear *integrable* Hamiltonian translation-invariant equations (KdV, etc.) in the works of M. J. Ablowitz, H. Segur, W. Eckhaus, A. van Harten, and others [53, 59].

For *nonintegrable* equations the global attraction to solitons (12) was established for the first time in [54]–[57] for translation-invariant systems of the wave and Maxwell equations coupled to a charged relativistic particle. The result of [55] gives the first rigorous proof of the *radiation damping* for the translation-invariant system of classical electrodynamics.

The proofs in [54] and [55] rely on a canonical transformation to the comoving frame and variational properties of solitons, as well as on the relaxation of the acceleration (9) under the Wiener condition for the particle charge density.

The multisoliton asymptotics (13) for *nonintegrable equations* were observed numerically in [58] in the case of 1D *relativistic* nonlinear wave equations.

The results of [54] and [58] are presented with details in Chapters 2 and 6, respectively.

3. Equations with the unitary symmetry group $G = U(1)$ Let us consider for example the case of the simplest representation

$$[T(e^{i\theta})\psi](x) := e^{i\theta}\psi(x), \qquad x \in \mathbb{R}^n \tag{14}$$

for $\theta \in R$. Then the asymptotics (6) means the *single-frequency asymptotics*

$$\psi(x,t) \sim \psi_\pm(x)e^{-i\omega_\pm t}, \qquad t \to \pm\infty, \tag{15}$$

where $\omega_\pm \in \mathbb{R}$.

Example 3 For the case of the coupled Maxwell–Schrödinger equations (8.2.1) with the symmetry group $U(1)$, the conjecture (6) reduces to the asymptotics (8.2.8) similar to (15).

The asymptotics (15) also means the global attraction to the solitary manifold formed by all *stationary orbits* which are solutions of type $\psi_\omega(x)e^{-i\omega t}$. The asymptotics are expected in the local seminorms $L^2(|x| < R)$ with any $R > 0$. The global attractor is a smooth manifold formed by the circles which are the orbits of the action of the symmetry group $U(1)$ (see Figure 2).

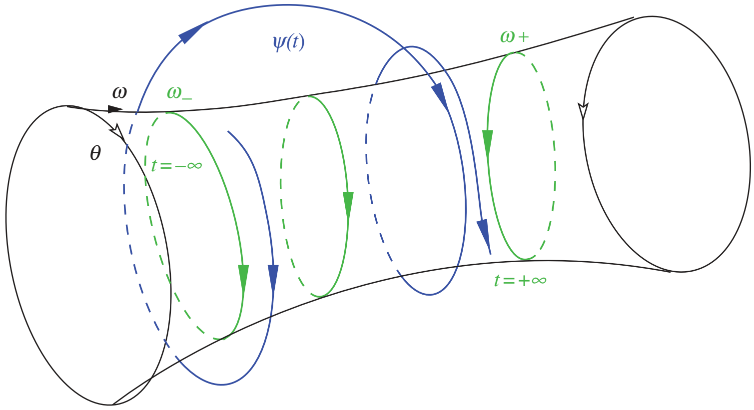

Figure 2 Convergence to stationary orbits.

Such an attraction *in local seminorms* $L^2(|x| < R)$ was proved (1) in [61]–[67] for the Klein–Gordon and Dirac equations coupled to a $U(1)$-invariant nonlinear oscillator; (2) in [60], for discrete approximations of such coupled systems, i.e., for the corresponding difference schemes; and (3) in [69]–[71] for the wave, Klein–Gordon, and Dirac equations with concentrated nonlinearities. More precisely, we have proved global attraction to the *solitary manifold* of all stationary orbits, though global attraction to particular stationary orbits, with fixed $\omega_\pm$, is still an open problem.

All these results were proved under the assumption that the equations are "strictly nonlinear." For linear equations, the global attraction obviously fails if the discrete spectrum consists of at least two different eigenvalues.

The proofs of these results rely on (1) the concept of omega-limit trajectory, (2) a nonlinear analog of the Kato theorem on the absence of embedded eigenvalues, (3) new theory of multipliers in the space of quasimeasures, and (4) novel application of the Titchmarsh convolution theorem. The results of [62]–[64] are presented with details in Chapter 3.

Existence and orbital stability of stationary orbits The existence of solutions $e^{\hat{\lambda}t}\Psi$ (*stationary G-orbits*) for G-invariant nonlinear wave equations (2) in the cases $G = U(1)$ and $G = \mathbb{R}^n$ was extensively studied in the 1960s–1980s. The most general results were obtained by W. Strauss, H. Berestycki, and P.-L. Lions [24, 25, 30]. M. Esteban, V. Georgiev, and E. Séré constructed in [27] stationary orbits for the relativistic nonlinear Maxwell–Dirac system (8.2.7) and for the Klein–Gordon–Dirac system. The key role in these papers was played by the Lusternik–Schnirelmann theory of critical points [28, 29].

In [26] G. M. Coclite and V. Georgiev constructed stationary orbits for the nonlinear Maxwell–Schrödinger system with the external Coulomb potential.

General theory of *orbital stability* of stationary G-orbits was developed by M. Grillakis, J. Shatah, and W. Strauss in [100, 101].

4. Equations with the orthogonal symmetry group $G = SO(3)$ For such generic equations, the asymptotics (6) means that

$$\psi(x,t) \sim e^{-i\hat{\Omega}_\pm t}\psi_\pm(x), \qquad t \to \pm\infty, \tag{16}$$

where $\hat{\Omega}_\pm$ are suitable representations of real skew-symmetric 3×3 matrices $\Omega_\pm \in \mathfrak{so}(3)$. This means that global attraction to "stationary $SO(3)$-orbits" occurs. Such asymptotics are proved in [88] for the Maxwell–Lorentz equations with rotating particle.

Generic equations Let us emphasize that, for example, we are conjecturing asymptotics (15) for *generic* $U(1)$*-invariant equations*. This means that the long-time behavior of solutions may be quite different for $U(1)$-invariant equations of "positive codimension." In particular, for solutions of the linear Schrödinger equation

$$i\dot{\psi}(x,t) = -\Delta\psi(x,t) + V(x)\psi(x,t), \qquad x \in \mathbb{R}^n,$$

the asymptotics (15) generally fail. Namely, any finite-energy solution admits the spectral representation

$$\psi(x,t) = \sum C_k\psi_k(x)e^{-i\omega_k t} + \int_0^\infty C(\omega)\psi(\omega,x)e^{-i\omega t}\,d\omega,$$

where ψ_k and $\psi(\omega,\cdot)$ are the corresponding eigenfunctions of the discrete and continuous spectrum, respectively. The last integral is a dispersive wave, which decays to zero in the norms $L^2(|x| < R)$ with any $R > 0$ (under appropriate conditions on the potential $V(x)$). Correspondingly, the attractor is the linear span of the eigenfunctions ψ_k. Thus, the long-time asymptotics does not reduce to a single term like (15), so the linear case is degenerate in this sense. Note that all our results [61]–[67] are established for a *strictly nonlinear case* (see the condition (3.1.16)), which eliminates linear equations).

Higher symmetry groups For more sophisticated symmetry groups $G = U(N)$, the asymptotics (6) mean the global attraction to N-frequency trajectories, which can be quasi-periodic. In particular, the symmetry groups $SU(2)$, $SU(3)$, and others were suggested in 1961 by M. Gell-Mann and Y. Ne'eman for strong interaction of baryons [222, 224]. This theory provides empirical evidence for the asymptotics (6), see Section 3.11.

On relations with Soffer's conjectures Note that our conjecture (6) specifies the concept of *localized solution/coherent structures* from the "Grande Conjecture" and the "Petite Conjecture" of A. Soffer (see [161], p. 460) in the context of the Banach spaces. The Grande Conjecture was proved in [47] for the case of a 1D wave equation coupled to a nonlinear oscillator (1.2.1). Moreover, suitable versions of the Grande Conjecture were also proved in [57, 88] for the 3D wave and Klein–Gordon and Maxwell equations coupled to a relativistic particle with sufficiently small charge (2.2.1) (see Remark 2.2.1). Finally, for any matrix symmetry group G, the asymptotics (6) corresponds to the Petite Conjecture since then the localized solutions $e^{g_\pm t}\psi_\pm(x)$ are quasi-periodic.

In this book we present available results on the global attraction (7)–(16) and related numerical experiments. Moreover, we survey the results on asymptotic stability of solitons and their adiabatic effective dynamics, on the dispersive decay and relations to quantum mechanics.

Asymptotic stability of solitons More precisely, we should phrase "asymptotic stability of solitary manifolds," which means a local attraction, i.e., for states sufficiently close to the manifold. There is a huge body of literature on this subject. In Chapter 4 we review the results on such local attraction that were developed in a series of articles [162]–[170] by V.S. Buslaev, G. Perelman, A. Soffer, D. Stuart, C. Sulem, T. P. Tsai, M. Weinstein, H. T. Yau, and others.

The crucial peculiarity of this attraction is the instability of the dynamics *along the solitary manifold*. This follows directly from the fact that solitons move with different speeds and therefore run away for large times. Analytically, this instability is caused by the presence of the eigenvalue $\lambda = 0$ in the spectrum of the generator of linearized dynamics. Namely, the tangent vectors to the solitary manifold are eigenvectors and associated vectors of the generator. They correspond to zero eigenvalue. Respectively, the Lyapunov theory is not applicable to this case.

This is why in the articles [162]–[169] an original strategy was developed for proving asymptotic stability of solitary manifolds. This strategy allows one to separate the unstable motion along the solitary manifold and the attraction in transversal directions to this manifold.