ORTHOGONAL POLYNOMIALS IN THE SPECTRAL ANALYSIS OF MARKOV PROCESSES

In pioneering work in the 1950s, S. Karlin and J. McGregor showed that the probabilistic aspects of certain Markov processes can be studied by analyzing the orthogonal eigenfunctions of associated operators. In the decades since, many authors have extended and deepened this surprising connection between orthogonal polynomials and stochastic processes.

This book gives a comprehensive analysis of the spectral representation of the most important one-dimensional Markov processes, namely discrete-time birth–death chains, birth–death processes and diffusion processes, and brings together all the main results from the extensive literature on the topic with detailed examples and applications. It also features an introduction to the basic theory of orthogonal polynomials and has a selection of exercises at the end of each chapter. The book is suitable for graduate students with a solid background in stochastic processes as well as researchers in orthogonal polynomials and special functions who want to learn about applications of their work to probability.

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Orthogonal Polynomials in the Spectral Analysis of Markov Processes

Birth–Death Models and Diffusion

MANUEL DOMÍNGUEZ DE LA IGLESIA
Universidad Nacional Autónoma de México

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To my wife Diana and my son Jorge

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Preface

The connection between stochastic processes, special functions and orthogonal polynomials has a long history. From the 1930s N. Wiener and later K. Itô knew about the connection between Hermite polynomials and integration theory with respect to Brownian motion. Around the 1950s many authors like M. Kac [80], W. Feller [53]–[56], E. Hille [71], W. Ledermann and G. E. Reuter [111], J. F. Barrett and D. G. Lampard [6], S. Karlin and J. McGregor [82]–[89], H. P. McKean [116] and D. G. Kendall [103] established an important connection between the transition probability functions of diffusion processes, continuous-time birth–death processes and discrete-time birth–death chains (in this order) by means of a spectral representation. This spectral representation is based on the spectral analysis of the infinitesimal operator associated with these special types of Markov processes and many probabilistic aspects can be analyzed in terms of the corresponding orthogonal eigenfunctions and eigenvalues. In the following years these relationships were developed further, finding connections with other stochastic processes like random matrices, Sheffer systems, Lévy processes, stochastic integration theory or Stein's method. For a brief account of all these relations see [129].

The main goal of this monograph is to give a comprehensive analysis of the main results related to the spectral representation of the most important one-dimensional Markov processes, namely discrete-time birth–death chains (also called random walks in some references, see [87]), birth–death processes and diffusion processes. Since the pioneering work of S. Karlin and J. McGregor in the 1950s, many authors have contributed to finding more applications of the spectral representation of the transition probability functions of these processes. This monograph tries to gather all the important results that appear in many publications over the last 60 years in one common text. The contents of this monograph served as a one-semester graduate advanced course taught at the Instituto de Matemáticas of the Universidad Nacional Autónoma de México in Fall 2018. The interested audience can be divided into two categories. On the one hand, it is intended for graduate students who have a solid background in the field of stochastic processes but are not so familiar with the theory of special functions and orthogonal polynomials. This monograph will give them alternative methods and ways of studying basic Markov processes by spectral methods. On the other hand, the book may also serve for students or researchers who are familiar with the theory of special functions and orthogonal polynomials but want to learn more about the connection between basic Markov processes and orthogonal polynomials.

In the experience of the author, graduate students are typically more familiar with probability theory and stochastic processes. This is the reason why an introduction to orthogonal polynomials is included in Chapter 1. This chapter also includes the concept of the *Stieltjes transform* and some of its properties, which will play a very important role in the spectral analysis of discrete-time birth–death chains and birth–death processes. A section about the spectral theorem for orthogonal polynomials (or *Favard's theorem*) will give insights about the relation between tridiagonal Jacobi matrices and spectral probability measures. We will focus then on the *classical families of orthogonal polynomials*, both of a continuous and a discrete variable. These families are characterized by the fact that they are eigenfunctions of a second-order differential or difference operator of hypergeometric type solving certain *Sturm–Liouville problems*. These classical families are part of the so-called *Askey scheme*.

In Chapter 2 we will perform the spectral analysis of discrete-time birth–death chains on $\mathbb{N}_0$, which are the most basic and important discrete-time Markov chains. These chains are characterized by a tridiagonal one-step transition probability matrix. We will obtain the so-called *Karlin–McGregor integral representation formula* of the n-step transition probability matrix in terms of orthogonal polynomials with respect to a probability measure ψ with support inside the interval $[-1,1]$. We will give an extensive collection of examples related to orthogonal polynomials, including gambler's ruin, the Ehrenfest model, the Bernoulli–Laplace model and the Jacobi urn model. The chapter ends with applications of the Karlin–McGregor formula to probabilistic aspects of discrete-time birth–death chains, such as recurrence, absorption, the strong ratio limit property and the limiting conditional distribution. Finally we will apply spectral methods to discrete-time birth–death chains on $\mathbb{Z}$, which are not so much studied in the literature.

In Chapter 3 we will perform the spectral analysis of birth–death processes on $\mathbb{N}_0$, which are the most basic and important continuous-time Markov chains. In this case, these processes will be characterized by an infinitesimal operator, which is a tridiagonal matrix whose spectrum is inside the interval $(-\infty,0]$. Again, we will obtain the *Karlin–McGregor integral representation formula* of the transition probability functions of the process in terms of orthogonal polynomials with respect to a probability measure ψ with support inside the interval $[0,\infty)$. Although many of the results are similar or equivalent to those of discrete-time birth–death chains, the methods and techniques are quite different. For instance, in this chapter, we will have to prove that the Karlin–McGregor representation formula is in fact a transition probability function of a birth–death process, something that was not necessary for the case of discrete-time birth–death chains. We will also provide an extensive collection of examples related to orthogonal polynomials, including the $M/M/k$ queue with $1 \leq k \leq \infty$ servers, the continuous-time Ehrenfest and Bernoulli–Laplace urn models, a genetics model of Moran and linear birth–death processes. As in the case of discrete-time birth–death chains, we will apply the Karlin–McGregor formula to probabilistic aspects of birth–death processes, such as processes with killing, recurrence, absorption, the strong ratio limit property, the limiting conditional distribution, the decay parameter, quasi-stationary distributions and bilateral birth–death processes on $\mathbb{Z}$.

In Chapter 4 we will perform the spectral analysis of one-dimensional diffusion processes, which are the most basic and important continuous-time Markov processes where now the state space is a continuous interval contained in $\mathbb{R}$. Diffusion processes are characterized by an infinitesimal operator, which is a second-order differential operator with a drift and a diffusion coefficient. We will obtain a spectral representation of the transition probability density of the process in terms of the orthogonal eigenfunctions of the corresponding infinitesimal operator, for which we will have to solve a *Sturm–Liouville problem* with certain boundary conditions. An analysis of the behavior of these boundary points will also be made. We will also give an extensive collection of examples related to special functions and orthogonal polynomials, including Brownian motion with drift and scaling, the Orstein–Uhlenbeck process, a population growth model, the Wright–Fisher model, the Jacobi diffusion model and the Bessel process, among others. Finally, we will study the concept of quasi-stationary distributions, for which the spectral representation will play an important role.

I would like to thank F. Alberto Grünbaum for introducing me to the fascinating connection between orthogonal polynomials and Markov processes. Back in 2009 I was visiting him as an undergraduate student at the University of California, Berkeley and we were studying one example of matrix-valued orthogonal polynomials coming from group representation theory which had a nice interpretation in terms of two-dimensional Markov chains. This was my first connection to the subject that brought me to write this monograph. I would also like to thank Eric A. van Doorn for reading the manuscript and providing an important list of corrections and additional material to include in the book. Unfortunately he tragically passed away before being

able to read the final version of this book. In closing I would like to thank the staff at Cambridge University Press for their support and cooperation during the preparation of this book.

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