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Excerpt
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Part I

Continuum Equations of Fluid Mechanics

1 Introduction

This book considers the mechanics of a *fluid*, defined as a material that continuously deforms under the influence of an applied shear stress, as depicted in Fig. 1.1. Here the fluid, initially at rest, lies between a stationary wall and a moving plate. Nearly all common fluids stick to solid surfaces. Thus, at the bottom, the fluid remains at rest; at the top, it moves with the velocity of the plate. The vectors indicate the fluid displacement, a distance that grows with time. At early time, the displacement profile varies nonlinearly with distance from the stationary surface. At later times, the displacement profile becomes linear. For nearly all common fluids, it is observed that a nonzero shear stress is required to maintain this motion. As configured, this fluid will never come to rest. Such a definition allows both liquids and gases to be considered fluids. In contrast, a solid will deform but relax to an equilibrium configuration when subjected to an applied shear stress.

Motion in response to *transverse* shear forces is fundamental to fluids and induces such behavior as fluid rotation in a long persisting *vortex* such as seen in weather patterns and aerodynamic applications. In such a swirling environment, fluid particles often veer far from neighboring fluid particles. In contrast, solid particles almost always retain the same particles as neighbors; except for rotation as a rigid body, there is no clear counterpart to a vortex in typical solids. Motion in response to *longitudinal* normal forces is also fundamental to fluids and can result in volumetric compression and expansion as well as acceleration in the direction of the net normal force. Solids respond to longitudinal normal forces in a similar manner; they may induce weaker volumetric compression and expansion and certainly acceleration in the direction of the net normal force.

We present an approach to *fluid mechanics* founded on the general principles of *rational continuum mechanics*. These general principles apply to all continuous materials: solids, liquids, and gases. There are many paths to understanding fluid mechanics, and good arguments can be

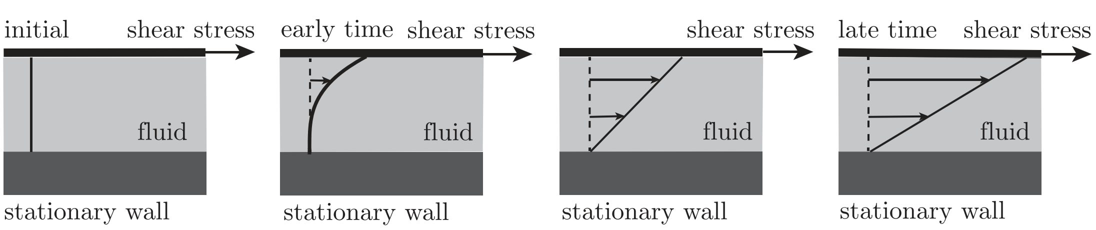

Figure 1.1 Diagram demonstrating the defining feature of a fluid: continuous deformation in response to an applied shear stress: snapshots of the fluid displacement profile at various times.

made for each. A typical first undergraduate class will combine a mix of basic equations, coupled with strong physical motivations, and allow the student to develop a knowledge that is of practical value, often driven by intuition. Such an approach works well within the confines of the intuition we develop in everyday life. It often fails when the engineer moves into unfamiliar territory. For example, lack of fundamental understanding of high Mach[1] number flows led to many aircraft and rocket failures in the 1950s. In such cases, a return to the formalism of a careful theory, one that clearly exposes the strengths and weaknesses of all assumptions, is invaluable in both understanding the true fluid physics, and applying that knowledge to engineering design.

Probably the most formal of approaches is that of the school of thought promoted by Truesdell,[2] who forcefully advocated for rational continuum mechanics. Truesdell developed a broadly based theory that encompassed all materials that could be regarded as continua, including solids, liquids, and gases, in the limit when the smallest volumes considered were sufficiently large so that the micro- and nanoscopic structure of these materials was unimportant. For fluids (both liquid and gas), such length scales are often at or below the order of microns, while for solids, the scales may be smaller, depending on the type of molecular structure. The difficulty of the Truesdellian approach is that it is burdened with a difficult notation and tends to become embroiled in proofs and philosophy, which while ultimately useful, can preclude learning basic fluid mechanics in the time scale of the typical student. It is possible, however, to give a discussion that respects the approach of Truesdell while also providing both technical rigor and accessibility. For example, Thorne and Blandford (2017) give a nuanced, detailed, and useful exposition of fluid mechanics in the context of physics, geometry, and experiment that complements well the formalism of rational continuum mechanics.

Here, we will attempt to steer between the fallible pragmatism of undergraduate fluid mechanics and the harsh formalism of the Truesdellian school. The presentation will pay due homage to rational continuum mechanics and will be geared towards a basic understanding of fluid behavior. We shall first spend some time carefully developing the governing equations for a compressible viscous fluid. We shall then study representative solutions of these equations in a wide variety of physically motivated limits in order to understand how the basic evolution principles of mass, linear momenta, angular momenta, and energy, coupled with constitutive relations, influence the behavior of fluids. In the end, it is hoped the reader will have an enhanced appreciation of the abilities and limitations of deterministic continuum mathematical physics to predict basic fluid behavior.

1.1 Mechanics

Mechanics is the broad superset of the topic matter of this book. It is the science that seeks an explanation for the motion of bodies based upon models grounded in axioms. Axioms, as in geometry, are statements that cannot be proved; they are useful insofar as they give rise to

[1] Ernst Mach, 1838–1926, Viennese physicist and philosopher who worked in optics, mechanics, and wave dynamics, and developed fundamental ideas of inertia.
[2] Clifford Ambrose Truesdell III, 1919–2000, American continuum mechanician and natural philosopher.

results that are consistent with empirical observation. A hallmark of science has been the struggle to identify the smallest set of axioms that are sufficient to describe our universe. When we find an axiom to be inconsistent with observation, it must be modified or eliminated. A familiar example of this is the Michelson[3]–Morley[4] experiment, which motivated Einstein[5] to modify the Newtonian[6] axioms of mass and mechanical energy into an axiom for mass-energy. There are many subsets of mechanics, for example statistical mechanics, relativistic mechanics, quantum mechanics, continuum mechanics, fluid mechanics, or solid mechanics. Each has its own set of axioms; often in certain physical limits, the axioms of one framework relax to those of another framework. For example, as the deviation of the velocity from the speed of light increases, the more robust axioms of Einstein's relativistic mechanics relax to those of Newton's mechanics.

1.2 Rational Continuum Mechanics

Newton introduces the concept of *rational* mechanics in the preface to the original 1687 edition of his *Principia* to distinguish it from both geometry and other types of mechanics that were common in his era. There Newton (1999, p. 382, originally presented 1687) states:

> ...*geometry* is commonly used in reference to magnitude, and *mechanics* in reference to motion. In this sense *rational mechanics* will be the science, expressed in exact proportions and demonstrations, of the motions that result from any forces whatever and of the forces that are required for any motions whatever.

Early mechanicians, such as Newton, dealt primarily with point masses and finite collections of distinct particles. Such systems are the easiest to study, and it makes more sense to grasp the simple before the complex. The discipline that considers such systems is often referred to as *classical mechanics*. Mathematically, such systems are generally characterized by a finite number of ordinary differential equations, and the properties of each particle (e.g. position, velocity) are taken to be functions of time only.

Continuum mechanics, generally attributed to Euler,[7] considers instead an infinite number of particles, which is in fact easier to analyze than a large finite number of particles. In continuum mechanics, every physical property (e.g. velocity, density, pressure) is taken to be a function of both time and space. Infinitesimal property variation from point to point in space is permitted. While variations are generally continuous, finite numbers of surfaces of discontinuous property variation are allowed. This models, for example, the contact between one continuous body and another or a shock wave within a inviscid fluid. Point discontinuities are not allowed, however. Finite valued material properties are required. Mathematically, such systems are characterized

[3] Albert Abraham Michelson, 1852–1931, Prussian-born American physicist.
[4] Edward Williams Morley, 1838–1923, American physical chemist.
[5] Albert Einstein, 1879–1955, German and later American physicist who developed the theory of relativity and made fundamental contributions to quantum mechanics and Brownian motion in fluid mechanics.
[6] Sir Isaac Newton, 1642–1727, English physicist, mathematician, and chief figure of the scientific revolution. Developed calculus, theories of gravitation, motion of bodies, and optics.
[7] Leonhard Euler, 1707–1783, Swiss-born mathematician and physicist.

by a finite number of partial differential equations in which the properties of the continuum material are functions of both space and time. It is possible to show that a partial differential equation can be thought of as an infinite number of ordinary differential equations, so this is consistent with our model of a continuum as an infinite number of particles.

In Truesdell's exposition on continuum mechanics, he suggests the following hierarchy:

- *bodies* exist,
- bodies are assigned to *place*,
- *geometry* is the theory of place,
- change of place in *time* is the *motion* of the body,
- a description of the motion of a body is *kinematics*,
- motion is the consequence of *forces*,
- the study of forces on a body is *dynamics*, sometimes called *kinetics*.

We will adapt this hierarchy in our exposition.

The modifier "rational" was applied by Truesdell to continuum mechanics to distinguish the formal approach advocated by his school, from less formal, though mainly not irrational, approaches to continuum mechanics. Rational continuum mechanics is developed with tools similar to those that Euclid[8] used for his geometry and Newton for his physics: formal definitions, axioms, and theorems, all accompanied by careful language and proofs. One of the hallmarks of rational continuum mechanics is a distinction between material-independent axioms, such as mass, momenta, and energy principles, from material-specific relations such as the ideal gas law. This book will recognize such distinctions, all the while following a less formal, albeit still rigorous, approach. The extensive literature associated with rational continuum mechanics considers a broad range of topics, and nuances of some aspects of its axioms, not relevant for this book, are not universally accepted; see for example Woods (1982) or Müller (2007) and references within.

1.2.1 Notions from Newtonian Mechanics

The following are useful notions from Newtonian mechanics. Here we use Newtonian to distinguish our mechanics from Einsteinian relativistic mechanics. Newton himself did not study continuum mechanics; however, notions from his studies of the mechanics of discrete sets of point masses extend to the mechanics of continua.

Space is three-dimensional and independent of time. An *inertial frame* is a reference frame in which the laws of physics are invariant; further, a body in an inertial frame with zero net force acting upon it does not accelerate. A *Galilean*[9] *transformation* specifies how to transform from one inertial frame to another inertial frame moving at constant velocity relative to the original frame. If a second inertial frame has constant velocity $\mathbf{v}_o = u_o\mathbf{i} + v_o\mathbf{j} + w_o\mathbf{k}$ relative to the original inertial frame, the Galilean transformation $(x, y, z, t) \rightarrow (x', y', z', t')$ is as follows:

[8] Euclid, Greek geometer of profound influence.
[9] Galileo Galilei, 1564–1642, Pisa-born Italian astronomer, physicist, and developer of experimental methods, first employed a pendulum to keep time, builder and user of telescopes used to validate the Copernican view of the universe, developer of the principle of inertia and relative motion.

$$x' = x - u_o t, \quad y' = y - v_o t, \quad z' = z - w_o t, \quad t' = t. \tag{1.1}$$

This must be accompanied with a transformation of the velocities:

$$u' = u - u_o, \quad v' = v - v_o, \quad w' = w - w_o. \tag{1.2}$$

1.2.2 Continuum Fields

In contrast to a single particle or finite set of particles for which time is the only independent variable, a fluid is modeled as a *continuum field* with properties that depend on both time and space. The notion of a continuum is rooted in Newtonian calculus; an example of a continuum from mathematics is the set of real numbers. A real number $x \in \mathbb{R}^1$ is a member of the set of real scalars, $\mathbb{R}^1$. It is often considered to reside on the x axis. The x axis may be finely partitioned. At the edges of a particular partition, x may have the respective values of x_n and x_{n+1}. The essence of the continuum assumption is that no matter how fine the partition, there exists an intermediate value $\tilde{x}$ with $x_n \leq \tilde{x} \leq x_{n+1}$. Here x is a geometrical property. Ordered pairs of real numbers $(x, y)^T \in \mathbb{R}^2$ reside in a plane, and ordered triples of real numbers $(x, y, z)^T \in \mathbb{R}^3$, reside in a volume. Here T is the transpose operator, employed so that the ordered pairs and triples are column vectors, by standard convention. We can extend these notions to fluid properties.

Density, ρ, is a material property of a fluid describing the concentration of its mass m within a volume V. Consider a sequence of shrinking volumes $V_1 > V_2 > V_3, \ldots, V_n$, each containing a sequence of shrinking masses, $m_1 > m_2 > m_3, \ldots, m_n$, sketched in Fig. 1.2. We expect the mass to be a function of the volume. Define the average density $\overline{\rho}_n$ as

$$\overline{\rho}_n = \frac{m_n}{V_n}. \tag{1.3}$$

In a true continuum, as we let $n \to \infty$, we could define the density at P to be

$$\rho(P) = \lim_{n \to \infty} \frac{m_n}{V_n}, \tag{1.4}$$

and importantly, expect the limiting value $\rho(P)$ to be finite and smoothly approaching its limiting value as $n \to \infty$. In this limit, we expect V_n to approach the infinitesimal volume dV surrounding the point P and m_n to approach the infinitesimal value dm. This gives

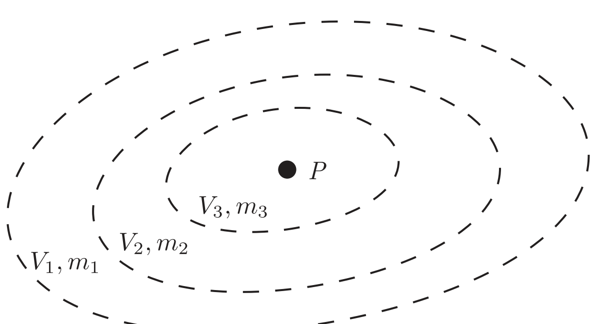

Figure 1.2 Diagram of sequence of volumes, each enclosing a respective mass, with the volume shrinking to a point P.

$$\rho = \frac{dm}{dV}. \tag{1.5}$$

Every point P in the continuum would possess a set of spatial coordinates and a local value of ρ at each point. One can also allow for time variation so $\rho = \rho(x, y, z, t)$. Then with $dV = dx\ dy\ dz$, one could integrate Eq. (1.5) over a finite volume to get

$$m(t) = \int_{z_1}^{z_2} \int_{y_1}^{y_2} \int_{x_1}^{x_2} \rho(x, y, z, t)\ dx\ dy\ dz. \tag{1.6}$$

In this book, properties such as $\rho(x, y, z, t)$ will be assumed to exist and contain all the features of a mathematical continuum, except at jumps associated with material interfaces or shock waves. This approach has proven to work well as long as the volumes being considered are sufficiently large to contain many fluid molecules.

Density is not used in classical Newtonian mechanics, as that discipline only considers point masses. Continuum mechanics will treat macroscopic effects only and ignore individual molecular effects. For example, molecules bouncing off a wall exchange momentum with the wall and induce pressure. We could use Newtonian mechanics for each particle collision to calculate the net wall force. Instead our approach amounts to considering the *average* over space and time of the net effect of millions of collisions on a wall.

1.2.3 Scalars, Vectors, and Tensors

We briefly introduce here the notion of fields composed of what are known as *scalars*, *vectors*, and *tensors*. This important topic will be considered in more detail in Section 2.1. A diagram depicting the nature of scalars, vectors, and tensors is given in Fig. 1.3.

Density is an example of a scalar property. A scalar property associates a single number with each point in time and space. *Scalars possess magnitude but not direction.* We can think of this by writing the usual notation

$$\rho = \rho(x, y, z, t), \tag{1.7}$$

indicating that ρ has functional variation with position and time. Other properties are not scalar, but are vector properties. For example, the velocity vector

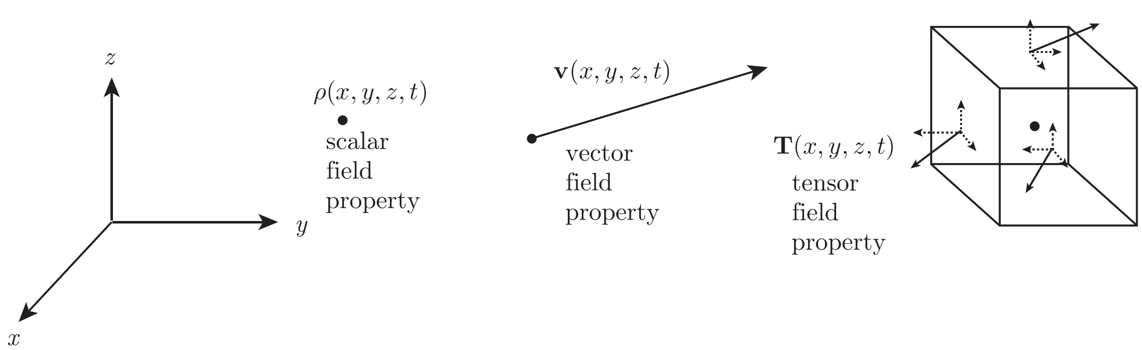

Figure 1.3 Diagram depicting the nature of scalars, vectors, and tensors.

$$\mathbf{v}(x,y,z,t) = u(x,y,z,t)\mathbf{i} + v(x,y,z,t)\mathbf{j} + w(x,y,z,t)\mathbf{k} = \begin{pmatrix} u(x,y,z,t) \\ v(x,y,z,t) \\ w(x,y,z,t) \end{pmatrix}, \tag{1.8}$$

associates three scalars u, v, w with each point in space and time. Here $\mathbf{i}$, $\mathbf{j}$, and $\mathbf{k}$ are the familiar set of orthonormal basis vectors associated with a Cartesian coordinate system. We will see that a vector can be characterized as a scalar associated with a particular direction in space; that is, *vectors possess both magnitude and direction.* Here we use a boldfaced notation for a vector. This is known as Gibbs[10] notation. We will study in Section 2.1 an alternate notation, developed by Einstein, known as Cartesian[11] index notation.

Other properties are not scalar or vector, but are what is known as tensors. The best known example is the *stress tensor*, whose physics and mathematics will be fully described in Section 4.2.2. One can think of a tensor as a quantity that associates a vector with a plane inclined at a selected angle passing through a given point in space. *Tensors possess magnitude, direction, and orientation relative to a plane.* For an infinitesimal cube surrounding a point, each of the faces of the cube can be associated with a unique vector. This is shown in Fig. 1.3, in which a distinct vector is associated with three orthogonal surfaces. Each vector on each surface is itself the sum of three orthogonal components. An example is the stress tensor $\mathbf{T}$. It can be thought of as associating three vectors (or nine scalars) with each spatial point. It is best expressed as a matrix:

$$\mathbf{T}(x,y,z,t) = \begin{pmatrix} T_{xx}(x,y,z,t) & T_{xy}(x,y,z,t) & T_{xz}(x,y,z,t) \\ T_{yx}(x,y,z,t) & T_{yy}(x,y,z,t) & T_{yz}(x,y,z,t) \\ T_{zx}(x,y,z,t) & T_{zy}(x,y,z,t) & T_{zz}(x,y,z,t) \end{pmatrix}. \tag{1.9}$$

The stress tensor will be important in the mechanics of fluids. It will describe, among other things, pressure forces normal to a surface and frictional forces tangential to a surface. It will be considered fully in Chapters 4 and 5. The set of spatial coordinates x, y, and z form a vector we call $\mathbf{x}$:

$$\mathbf{x} = x\mathbf{i} + y\mathbf{j} + z\mathbf{k} = \begin{pmatrix} x \\ y \\ z \end{pmatrix}. \tag{1.10}$$

Our scalar, vector, and tensor fields are typically functions of the vector $\mathbf{x}$ and time t, and may be compactly written in the form $\rho(\mathbf{x}, t)$, $\mathbf{v}(\mathbf{x}, t)$, $\mathbf{T}(\mathbf{x}, t)$.

1.3 Molecular Limits of Continuum Theory

Continuum theory fails in scenarios in which the length and time scales are of comparable magnitude to molecular scales. Important applications for which the continuum assumption is inappropriate include rarefied gas dynamics (relevant for low Earth orbit vehicles), and

[10] Josiah Willard Gibbs, 1839–1903, American mechanical engineer who made fundamental contributions to vector analysis, statistical mechanics, thermodynamics, and chemistry.

[11] René Descartes, 1596–1650, French mathematician and philosopher who developed analytic geometry.

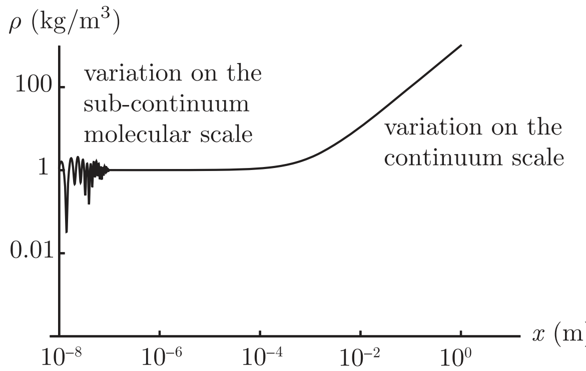

Figure 1.4 Diagram of possible density variation of a gas near atmospheric pressure as a function of length scale.

nanoscale heat transfer (relevant in cooling of computer chips). In commonly encountered physical scenarios, we expect the density to vary with distance on a macroscale, approach a limiting value at the microscale, and become ill-defined below a cutoff scale below which molecular effects are important. That is to say, when V_n from Fig. 1.2 becomes too small, such that only a few molecules are contained within it, we expect wild oscillations in ρ, and a unique value of ρ in the limit as $V_n \to 0$ formally does not exist.

To get some idea of the scales involved, we note that for air at atmospheric pressure and temperature, the time and distance between molecular collisions provide the limits of the continuum. Under these conditions, we observe for air that lengths $> 0.1\,\mu\text{m}$, and times $> 0.1\,\text{ns}$ will be sufficient to admit the continuum assumption. For denser gases, these cutoff scales are smaller. For lighter gases, these cutoff scales are larger. A depiction of a possible density variation in a gas near atmospheric pressure as a function of length scale is given in Fig. 1.4.

Details of collision theory can be found in texts such as that of Vincenti and Kruger (1965, pp. 12–26). The simplest model treats gases as composed of elastic spheres of diameter d moving within a volume that is mainly a vacuum. The model is valid in the limit in which the mean free path length λ between collisions is large relative to d. They show for air that λ is well modeled by:

$$\lambda = \frac{\mathcal{M}}{\sqrt{2}\pi \mathcal{N} \rho d^2}. \tag{1.11}$$

Here $\mathcal{M}$ is the molecular mass, $\mathcal{N}$ is Avogadro's number, and d is the molecular diameter, sometimes known as the *Lennard-Jones*[12] *diameter*.

Example 1.1 Find the variation of mean free path with density for air.

Solution

For a typical air mixture, the mixture molecular mass is taken as $\mathcal{M} = 28.97\,\text{kg/kmole}$. We turn to Vincenti and Kruger for other numerical parameter values: $\mathcal{N} = 6.022\,52 \times 10^{23}$ molecule/mole,

[12] John Edward Lennard-Jones, 1894–1954, British mathematician and physicist.