1

Calculus in Locally Convex Spaces

1.1 Introduction

It is well known that 'multidimensional calculus', aka 'Fréchet calculus', carries over to the realm of Banach spaces and Banach manifolds (see e.g. Lang, 1999). As we have seen in the Preface, Banach spaces are often not sufficient for our purposes. To generalise derivatives we will, as a minimum, need vector spaces with an amenable topology (which need not be induced by a norm).

1.1 Definition Consider a vector space E. A topology $\mathcal{T}$ on E making addition $+\colon E \times E \to E$ and scalar multiplication $\cdot\colon \mathbb{R} \times E \to E$ continuous is called a *vector topology* (where $\mathbb{R}$ carries the usual norm topology). We then say that $(E, \mathcal{T})$ (or E for short) is a *topological vector space* (or *TVS* for short).

1.2 Example (a) Every normed space and, in particular, every finite-dimensional vector space is a topological vector space.

(b) For a more interesting example, fix $0 < p < 1$. Two measurable functions $\gamma, \eta\colon [0,1] \to \mathbb{R}$ are equivalent $\gamma \sim \eta$ if and only if $\int_0^1 |\gamma(s) - \eta(s)| \mathrm{d}s = 0$. Denote by $L^p[0,1]$ the vector space of all equivalence classes $[\gamma]$ of functions such that $\int_0^1 |\gamma(s)|^p \mathrm{d}s < \infty$. Topologise $L^p[0,1]$ via the metric topology induced by

$$d([\gamma],[\eta]) := \int_0^1 |\gamma(s) - \eta(s)|^p \mathrm{d}s.$$

In a metric space, we can test continuity of the vector space operations using sequences. For this, pick $\lambda_n \to \lambda \in \mathbb{R}$ and $[\gamma_n] \to [\gamma], [\eta_n] \to [\eta]$ (with respect to d) and use the triangle inequality to obtain:

$$
\begin{aligned}
&d(\lambda_n[\gamma_n]+[\eta_n],\lambda[\gamma]+[\eta])\\
&\qquad\le|\lambda_n-\lambda|^p d([\gamma_n],[0])+|\lambda|^p d([\gamma_n],[\gamma])+d([\eta_n],[\eta]).
\end{aligned}
$$

This shows that the vector space operations are continuous, that is, $L^p[0,1]$ is a TVS.

In topological vector spaces, differentiable curves can be defined as follows:

1.3 Definition Let E be a topological vector space. A continuous mapping $\gamma\colon I\to E$ from a non-degenerate interval[1] $I\subseteq\mathbb{R}$ is called a C^0-curve. A C^0-curve is called a C^1-*curve* if the limit

$$\gamma'(s):=\lim_{t\to 0}\frac{1}{t}(\gamma(s+t)-\gamma(s))$$

exists for all $s\in I^\circ$ (interior of I) and extends to a continuous map $\frac{\mathrm{d}}{\mathrm{d}t}\gamma:=\gamma'\colon I\to E,\ s\mapsto\gamma'(s)$. Recursively for $k\in\mathbb{N}$, we call γ a C^k-*curve* if γ is a C^{k-1}-curve and $\frac{\mathrm{d}^{k-1}}{\mathrm{d}t^{k-1}}\gamma$ is a C^1-curve. Then $\frac{\mathrm{d}^k}{\mathrm{d}t^k}\gamma:=\left(\frac{\mathrm{d}^{k-1}}{\mathrm{d}t^{k-1}}\gamma\right)'$. If γ is a C^k-curve for every $k\in\mathbb{N}_0$, we also say that γ is *smooth* or of class C^∞.

Unfortunately, calculus on topological vector spaces is, in general, ill behaved. The next exercise shows that derivatives may fail to give us meaningful information.

Exercises

1.1.1 Given $0<p<1$ we let $L^p[0,1]$ be the topological vector space from Example 1.2(b). Recall that the topology on $L^p[0,1]$ is induced by the metric $d([\gamma],[\eta]):=\int_0^1|\gamma(s)-\eta(s)|^p\,\mathrm{d}s$. For a set $A\subseteq[0,1]$ write $\mathbf{1}_A$ for the characteristic function and define

$$\beta\colon[0,1]\to L^p[0,1],\quad \beta(t):=[\mathbf{1}_{[0,t[}].$$

Show that β is an injective C^1-curve with $\beta'(t)=0$, for all $t\in[0,1]$.

Obviously we would like to avoid this defect, and so we have to strengthen the assumptions on our vector spaces.

[1] That is, I has more than one point. In the following, we will always assume this when talking about intervals.

1.2 Curves in Locally Convex Spaces

Calculus in topological vector spaces exhibits pathologies that can be avoided by strengthening the requirements on the underlying space. This leads to locally convex spaces, whose topology is induced by so-called seminorms. See also Appendix A for more information on locally convex spaces.

1.4 Definition Let E be a vector space. A map $p\colon E \to [0,\infty[$ is called a *seminorm* if it satisfies the following:

(a) $p(\lambda x) = |\lambda| p(x), \forall \lambda \in \mathbb{R}, x \in E$,
(b) $p(x+y) \leq p(x) + p(y)$.

Note that, in contrast with the definition of a norm, we did not require that $p(x) = 0$ if and only if $x = 0$. The next definition uses the notion of an initial topology, which we recall for the reader's convenience in Appendix B.

1.5 Definition A topological vector space $(E, \mathcal{T})$ is called a *locally convex space* if there is a family $\{p_i : E \to [0,\infty[\mid i \in I\}$ of continuous seminorms for some index set I such that

(a) $\mathcal{T}$ is the initial topology with respect to the canonical projections $\{q_i : E \to E/p_i^{-1}(0)\}_{i \in I}$ onto the normed spaces $E/p_i^{-1}(0)$.
(b) If $x \in E$ with $p_i(x) = 0$ for all $i \in I$, then $x = 0$. Thus the seminorms separate the points, that is, $\mathcal{T}$ has the Hausdorff property.[2]

We then say that the topology $\mathcal{T}$ is *generated by the family of seminorms* $\{p_i\}_{i \in I}$ and call this family a *generating family of seminorms*. Usually we suppress $\mathcal{T}$ and write $(E, \{p_i\}_{i \in I})$ or simply E instead of $(E, \mathcal{T})$.

Alternative to (a) We will see in Appendix A that equivalent to (a), we can define $\mathcal{T}$ to be the unique vector topology determined by the basis of 0-neighbourhoods given by (finite) intersections of the balls $B_{i,\varepsilon}(0) = \{x \in E \mid p_i(x) < \varepsilon\}$, where p_i runs through a generating family of seminorms. These balls are all convex, thus justifying the name locally convex space.

A locally convex space $(E, \{p_i\}_{i \in \mathbb{N}})$ with a countable system of seminorms is *metrisable* (i.e. its topology is induced by a metric; see Exercise 1.2.1) and if E is complete, it is called *Fréchet space*.

1.6 Example (a) Every normed space $(E, \|\cdot\|)$ is a locally convex space, where the family of seminorms consists only of the norm $\|\cdot\|$.

[2] Some authors do not require separation of points, whence our locally convex spaces are Hausdorff locally convex spaces in their terminology.

(b) Consider the space $C^\infty([0,1],\mathbb{R})$ of all smooth functions from the interval $[0,1]$ to $\mathbb{R}$ (with pointwise addition and scalar multiplication). This space is not naturally a normed space.[3] We define a family of seminorms on it via

$$\|f\|_n := \sup_{0\leq k\leq n} \left\| \frac{\mathrm{d}^k}{\mathrm{d}t^k} f \right\|_\infty = \sup_{0\leq k\leq n} \sup_{t\in[0,1]} \left| \frac{\mathrm{d}^k}{\mathrm{d}t^k} f(t) \right|, n \in \mathbb{N}_0.$$

The topology generated by the seminorms is called the *compact-open C^∞-topology* and turns $C^\infty([0,1],\mathbb{R})$ into a locally convex space, which is even a Fréchet space (Exercise 1.2.2).

Locally convex spaces have many good properties, for example, they admit enough continuous linear functions to separate the points, that is, the following holds.

1.7 Theorem (Hahn–Banach (Meise and Vogt, 1997, Proposition 22.12)) *For a locally convex space E the continuous linear functionals separate the points, that is, for each pair $x,y \in E$ there exists a continuous linear $\lambda\colon E \to \mathbb{R}$ such that $\lambda(x) \neq \lambda(y)$.*

1.8 Definition Let E be a locally convex space, then we denote by $E' = L(E,\mathbb{R})$ the continuous linear maps from E to $\mathbb{R}$. The space E' is the so-called *dual space* of E. There are several ways to turn E' into a locally convex space (Rudin, 1991, p. 63f) but, in general, we will not need a topology beyond the special case if E is a Banach space and E' carries the operator norm topology.

With the help of the Hahn–Banach theorem, we can avoid the pathologies observed for topological vector spaces. To this end, we need the notion of a weak integral.

1.9 Definition Let $\gamma\colon I \to E$ be a C^0-curve in a locally convex space E and $a,b \in I$. If there exists $z \in E$ such that

$$\lambda(z) = \int_a^b \lambda(\gamma(t))\mathrm{d}t, \quad \forall \lambda \in E',$$

then $z \in E$ is called the *weak integral of* γ from a to b and denoted $\int_a^b \gamma(t)\mathrm{d}t := z$.

Note that weak integrals (if they exist) are uniquely determined due to the Hahn–Banach theorem.

[3] For any normed topology, the differential operator $D\colon C^\infty([0,1],\mathbb{R}) \to C^\infty([0,1],\mathbb{R})$, $D(f) = f'$ must be discontinuous (which is certainly undesirable). To see this, recall that a continuous linear map on a normed space has bounded spectrum, but D has arbitrarily large eigenvalues (consider $f_n(t) := \exp(nt)$, $n \in \mathbb{N}$).

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1.10 Proposition (First part of the fundamental theorem of calculus) *Let $\gamma\colon I \to E$ be a C^1-curve in a locally convex space E and $a, b \in I$, then*

$$\gamma(b) - \gamma(a) = \int_a^b \gamma'(t)\mathrm{d}t.$$

Proof Let $\lambda \in E'$. It is easy to see that $\lambda \circ \gamma\colon I \to \mathbb{R}$ is a C^1-curve with $(\lambda \circ \gamma)' = \lambda \circ (\gamma')$. The standard fundamental theorem of calculus yields

$$\lambda(\gamma(b) - \gamma(a)) = \lambda(\gamma(b)) - \lambda(\gamma(a)) = \int_a^b (\lambda \circ \gamma)'(s)\mathrm{d}s = \int_a^b \lambda(\gamma'(s))\mathrm{d}s.$$

Hence $z = \gamma(b) - \gamma(a)$ satisfies the defining property of the weak integral. □

Note that Proposition 1.10 implies that $L^p[0,1]$ cannot be a locally convex space for $0 < p < 1$; see Rudin (1991, 1.47) for an elementary proof of this fact.

1.11 Remark Also the second part of the fundamental theorem of calculus is true in our setting. Thus if $\gamma\colon I \to E$ is a C^0-curve, $a \in I$ and the weak integral

$$\eta(t) := \int_a^t \gamma(s)\mathrm{d}s$$

exists for all $t \in I$. Then $\eta\colon I \to E$ is a C^1-curve in E, and $\eta' = \gamma$.

The proof, however, needs more techniques based on convex sets which we do not wish to go into (see Glöckner and Neeb, forthcoming).

The reader may wonder now, when do weak integrals of curves exist? One can prove that weak integrals of continuous curves always exist *in the completion of a locally convex space*. The key point is that the integrals can be defined using Riemann sums, but these do not necessarily converge in the space itself (Kriegl and Michor, 1997, Lemma 2.5). Thus weak integrals exist for suitably complete spaces. To avoid getting bogged down with the discussion of completeness properties, we define the following:

1.12 Definition A locally convex space E is *Mackey complete* if for each smooth curve $\gamma\colon [0,1] \to E$ there exists a smooth curve $\eta\colon [0,1] \to E$ with $\eta' = \gamma$.

Due to the fundamental theorem of calculus this implies that $\eta(s) - \eta(0) = \int_0^s \gamma(t)\mathrm{d}t$. Thus the weak integral of smooth curves exists in Mackey complete spaces.

1.13 Remark Mackey completeness is a very weak completeness condition, in particular, sequential completeness (i.e. Cauchy sequences converge in the space) implies Mackey completeness. This is evident from the

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alternative characterisation of Mackey completeness using sequences; see Definition A.1. Note, however, that it is not entirely trivial to find examples of Mackey complete but not sequentially complete spaces. We mention here that the space $\mathcal{K}(E,F)$ of compact operators between two (infinite-dimensional) Banach spaces E,F with the strong operator topology is not sequentially complete but Mackey complete (see Voigt, 1992).

However, in metrisable locally convex spaces (e.g. in normed spaces) Mackey completeness is equivalent to completeness; see Jarchow (1981, 10.1.4). We refer to Kriegl and Michor (1997, I.2) for more information on Mackey completeness. In particular, Kriegl and Michor (1997, Theorem 2.14) show that integrals exist for C^1-curves in Mackey complete spaces.

So far we have defined differentiable curves with values in locally convex spaces. The next step is to consider differentiable mappings between locally convex spaces. Here a different notion of calculus is needed. It turns out that (even on Fréchet spaces) there are many generalisations of Fréchet calculus (see Keller, 1974) without a uniquely preferable choice. In the next section, we present a simple and versatile notion called Bastiani calculus. Another popular approach to calculus in locally convex spaces, the so-called convenient calculus, is discussed in Appendix A.7.

Exercises

1.2.1 Let $(E,\{p_n\}_{n\in\mathbb{N}})$ be a locally convex space whose topology is generated by a countable set of seminorms. Prove that

$$d(x,y) := \sum_{n\in\mathbb{N}} 2^{-n} \tfrac{p_n(x-y)}{p_n(x-y)+1}$$

is a metric on E and the metric topology coincides with the locally convex topology.

1.2.2 Consider $C^\infty([0,1],\mathbb{R})$ with the compact open C^∞-topology (see Example 1.6).

(a) Show that a sequence $(f_k)_{k\in\mathbb{N}}$ converges to f in this topology if and only if for all $\ell \in \mathbb{N}_0$ $\left(\frac{\mathrm{d}^\ell}{\mathrm{d}t^\ell} f_k\right)_k$ converges uniformly to $\frac{\mathrm{d}^\ell}{\mathrm{d}t^\ell} f$.
Hint: The uniform limit of a sequence of continuous functions is continuous. If a function sequence and the sequence of (first) derivatives converges, the limit of the sequence is differentiable.

(b) Deduce that every Cauchy sequence in the compact open C^∞-topology converges to a smooth function. As $C^\infty([0,1],\mathbb{R})$ is a metric space by Exercise 1.2.1, this implies that the space is complete, that is, a Fréchet space.

(c) Show that the differential operator

$$D\colon C^\infty([0,1],\mathbb{R}) \to C^\infty([0,1],\mathbb{R}), \qquad f \mapsto f'$$

is continuous linear. *Hint:* Lemma A.5.

1.2.3 Let $(E, \{p_i\}_I)$ be a locally convex space whose topology is generated by a *finite* set of seminorms. Show that $p(x) = \max_{i\in I} p_i(x)$ defines a norm on E, which induces the same topology as the family $\{p_i\}$. In this case we call E normable.

1.2.4 Establish the following properties of weak integrals:

(a) If the weak integrals of $\gamma, \eta\colon [\mathrm{a},\mathrm{b}] \to E$ from a to b exist and $s \in \mathbb{R}$, then also the weak integral of $\gamma + s\eta$ exists and $\int_a^b (\gamma(t) + s\eta(t))\mathrm{d}t = \int_a^b \gamma(t)\mathrm{d}t + s\int_a^b \eta(t)\mathrm{d}t$.

(b) If $\gamma\colon [\mathrm{a},\mathrm{b}] \to E$ is constant, $\gamma(t) \equiv K$, then $\int_a^b \gamma(t)\mathrm{d}t$ exists and equals $(b-a)K$.

(c) $\int_a^c \gamma(t)\mathrm{d}t = \int_a^b \gamma(t)\mathrm{d}t + \int_b^c \gamma(t)\mathrm{d}t$ (if the integrals exist).

1.2.5 Let $\gamma\colon I \to E$ be a C^k-curve ($k \in \mathbb{N}$) and $\lambda\colon E \to F$ be continuous linear for E, F locally convex. Show that $\lambda \circ \gamma$ is C^k such that $\frac{\mathrm{d}^\ell}{\mathrm{d}t^\ell}(\lambda\circ\gamma) = \lambda \circ \left(\frac{\mathrm{d}^\ell}{\mathrm{d}t^\ell}\gamma\right)$, $1 \le \ell \le k$.

1.2.6 Endow a vector space E with a topology $\mathcal{T}$ generated by seminorms as in Definition 1.5. Show that $(E, \mathcal{T})$ is a topological vector space (and so requiring that locally convex spaces are topological vector spaces was superfluous).

1.3 Bastiani Calculus

Bastiani calculus (also called Keller's C^k_c-theory; Keller, 1974), introduced in Bastiani (1964), builds a calculus around directional derivatives and their continuity. It is the basis of our investigation as this calculus works in locally convex spaces beyond the Banach setting.

1.14 Definition Let E, F be locally convex spaces, $U \Subset E$, $f\colon U \to F$ a map and $r \in \mathbb{N}_0 \cup \{\infty\}$. If it exists, we define for $(x,h) \in U \times E$ the *directional derivative*

$$df(x;h) \coloneqq D_h f(x) \coloneqq \lim_{\mathbb{R}\setminus\{0\}\ni t\to 0} t^{-1}(f(x+th) - f(x)).$$

We say that f is C^r if the *iterated directional derivatives*

$$d^k f(x; y_1, \ldots, y_k) := (D_{y_k} D_{y_{k-1}} \cdots D_{y_1} f)(x)$$

exist for all $k \in \mathbb{N}_0$ such that $k \leq r$, $x \in U$ and $y_1, \ldots, y_k \in E$ and define continuous maps $d^k f \colon U \times E^k \to F$ (where $d^0 f := f$). If f is C^k for all $k \in \mathbb{N}_0$ we say that f is *smooth* or C^∞. Note that $df = d^1 f$ and for curves $c \colon I \to E$ we have $c'(t) = dc(t; 1)$.

1.15 Remark Note that the iterated directional derivatives are only taken with respect to the first variable (i.e. of the map $x \mapsto df(x; v)$, where v is supposed to be fixed). One can alternatively define iterated differentials to derivate with respect to all variables, but this leads to the same differentiability concept (see Glöckner, 2002 for a detailed explanation). The following observations are easily proved from the definitions:

(a) $d^2 f(x; v, w) = \lim_{t \to 0} t^{-1}(df(x + tw; v) - df(x; v))$.
(b) $d^k f(x; v_1, \ldots, v_k) = \frac{\mathrm{d}}{\mathrm{d}t}\Big|_{t=0} d^{k-1} f(x + tv_k; v_1, \ldots, v_{k-1})$.
(c) f is C^k if and only if f is C^{k-1} and $d^{k-1} f$ is C^1. Then $d^k f = d(d^{k-1} f)$.

Finally, there is a version of the *Schwarz theorem* which states that the order of directions $v_1, \ldots, v_k$ in $d^k f(x; v_1, \ldots, v_k)$ is irrelevant (see Exercise 1.3.3).

1.16 Example Let $A \colon E \to F$ be a continuous linear map between locally convex spaces. Then A is C^1, as we can exploit

$$dA(x; v) = \lim_{t \to 0} t^{-1}(A(x + tv) - A(x)) = \lim_{t \to 0} A(v) = A(v).$$

In particular, since A is continuous, so is the first derivative and we see that A is a C^1-map. Computing the second derivative, we use that the first derivative is constant in x (but not in v!) to obtain

$$\begin{aligned} d^2 A(x; v, w) &= D_w(dA(x; v)) = \lim_{t \to 0} t^{-1}(dA(x + tw; v) - dA(x; v)) \\ &= \lim_{t \to 0} t^{-1}(A(v) - A(v)) = 0. \end{aligned}$$

In conclusion A is a C^2-map (obviously even a C^∞-map) whose higher derivatives vanish.

1.17 Lemma *Let $f \colon E \supseteq U \to F$ be a C^1-map. Then $df(x; \cdot)$ is homogeneous, that is, $df(x; sv) = s\,df(x; v)$ for all $x \in U, v \in E$ and $s \in \mathbb{R}$.*

Proof As $df(x; 0v) = df(x; 0) = 0 = 0df(x; v)$, we may assume that $s \neq 0$ and thus $df(x; sv) = \lim_{t \to 0} t^{-1}(f(x + tsv) - f(x)) = s \lim_{t \to 0}(st)^{-1}(f(x + tsv) - f(x)) = s\,df(x; v)$. □

1.18 Proposition (Mean value theorem on locally convex spaces) *Let E, F be locally convex spaces and $f\colon U \to F$ a C^1-map on $U \Subset E$. Then*

$$f(y) - f(x) = \int_0^1 df(x + t(y-x); y-x)\mathrm{d}t \qquad (1.1)$$

for all $x, y \in U$ such that U contains the line segment $\overline{xy} := \{tx + (1-t)y \mid t \in [0,1]\}$.

Proof Note that the curve $\gamma\colon [0,1] \to F, \gamma(t) := f(x + t(y-x))$ is differentiable at each $t \in [0,1]$. Its derivative is

$$\gamma'(t) = \lim_{s\to 0} s^{-1}(\gamma(t+s) - \gamma(t)) = df(x + t(y-x), y-x),$$

whence γ' is continuous (as df is) and thus a C^1-curve. Apply now the Fundamental theorem 1.10 to γ' to obtain (1.1). □

On a locally convex space, every point has arbitrarily small convex neighbourhoods. Convex neighbourhoods contain all line segments between points in the neighbourhood, whence Proposition 1.18 is available on these neighbourhoods. As a consequence we obtain the following.

1.19 Corollary *If $f\colon U \to F$ is a C^1-map with $df \equiv 0$, then f is locally constant.*

Proof For $x \in U$ choose a convex neighbourhood $x \in V \subseteq U$ (see Appendix A). For each $y \in V$ the line segment connecting x and y is contained in V, and so the vanishing of the derivative with (1.1) implies $f(x) = f(y)$ and f is constant on V. □

1.20 Proposition (Rule on partial differentials) *Let E_1, E_2, F be locally convex spaces, $U \Subset E_1 \times E_2$ and let $f\colon U \to F$ be continuous. Then f is C^1 if and only if the limits*

$$d_1 f(x,y;v_1) := \lim_{t\to 0} t^{-1}(f(x+tv_1, y) - f(x,y)),$$
$$d_2 f(x,y;v_2) := \lim_{t\to 0} t^{-1}(f(x, y+tv_2) - f(x,y))$$

exist for all $(x,y) \in U$ and $(v_1, v_2) \in E_1 \times E_2$ and extend to continuous mappings $d_i f\colon U \times E_i \to F, i = 1,2$. In this case,

$$df(x,y;v_1,v_2) = d_1 f(x,y;v_1) + d_2 f(x,y;v_2), \quad \forall (x,y) \in U, (v_1,v_2) \in E_1 \times E_2. \qquad (1.2)$$

Proof If f is C^1 the mappings $d_i f$ clearly exist and are continuous. Conversely, let us assume that the mappings $d_i f$ exist and are continuous. For

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$(x,y) \in U$, $(v_1,v_2) \in E \times F$, we fix $\varepsilon > 0$ such that $(x,y) + t(v_1,v_2) \in U$ whenever $|t| < \varepsilon$. Now if we fix the ith component of f we obtain a C^1-mapping (by hypothesis, the derivative is $d_i f$). Therefore Proposition 1.18 with Lemma 1.17 yields

$$\begin{aligned}\frac{f((x,y)+t(v_1,v_2)) - f(x,y)}{t} &= \frac{f(x+tv_1, y+tv_2) - f(x+tv_1,y)}{t} + \frac{f(x+tv_1,y)-f(x,y)}{t} \\ &= \int_0^1 d_2 f(x+tv_1, y+stv_2; v_2)\mathrm{d}s + \int_0^1 d_1 f(x+stv_1, y; v_1)\mathrm{d}s. \qquad (1.3)\end{aligned}$$

The integrals (1.3) make sense also for $t = 0$, whence they define maps $I_i \colon]-\varepsilon,\varepsilon[\to H$. Due to continuous dependence on the parameter t,[4] the right-hand side of (1.3) converges for $t \to 0$. We deduce that the limit df exists and satisfies (1.2) which is continuous, whence f is C^1. □

The following alternative characterisation of C^1-maps will turn the proof of the chain rule into a triviality. However, we shall only sketch the proof to avoid discussing convergence issues of the weak integral involved.

1.21 Lemma *A map $f \colon E \supseteq U \to F$ is of class C^1 if and only if there exists a continuous mapping, the difference quotient map,*

$$f^{[1]} \colon U^{[1]} := \{(x,v,s) \in U \times E \times \mathbb{R} \mid x + sv \in U\} \to F$$

such that $f(x+sv) - f(x) = sf^{[1]}(x,v,s)$ for all $(x,v,s) \in U^{[1]}$.

Proof Let us assume first that $f^{[1]}$ exists and is continuous. Note that $U^{[1]} \Subset U \times E \times \mathbb{R}$. Then $df(x;v) = f^{[1]}(x,v,0)$ exists and is continuous as a partial map of $f^{[1]}$. So f is C^1. Conversely, if f is C^1, the map

$$f^{[1]}(x,v,s) := \begin{cases} s^{-1}(f(x+sv) - f(x)), & (x,v,s) \in U^{[1]}, s \neq 0, \\ df(x;v), & (x,v,s) \in U^{[1]}, s = 0 \end{cases}$$

is continuous on the open set $U^{[1]} \setminus \{(x,v,s) \in U^{[1]} \mid s = 0\}$. That $f^{[1]}$ extends to a continuous map on all of $U^{[1]}$ follows from continuity of parameter-dependent weak integrals; see Bertram et al. (2004, Proposition 7.4) for details. □

1.22 Lemma *If $f \colon E \supseteq U \to F$ is C^1, then $df(x;\cdot) \colon E \to F$ is a continuous linear map for each $x \in U$.*

[4] We are cheating here; the continuous dependence of weak integrals on parameters has not been established in this book. See Hamilton (1982, I Theorem 2.1.5) for a proof that carries over to our setting.