Nonequilibrium Statistical Physics

Second Edition

Statistical mechanics is hugely successful when applied to physical systems at thermodynamic equilibrium; however, most natural phenomena occur in nonequilibrium conditions, and more sophisticated techniques are required to address this increased complexity. This second edition presents a comprehensive overview of nonequilibrium statistical physics, covering essential topics such as Langevin equations, Lévy processes, fluctuation relations, transport theory, directed percolation, kinetic roughening, and pattern formation. The first part of the book introduces the underlying theory of nonequilibrium physics, the second part develops key aspects of nonequilibrium phase transitions, and the final part covers modern applications. A pedagogical approach has been adopted for the benefit of graduate students and instructors, with clear language and detailed figures used to explain the relevant models and experimental results. With the inclusion of original material and organizational changes throughout the book, this updated edition will be an essential guide for graduate students and researchers in nonequilibrium thermodynamics.

Roberto Livi is an honorary professor of theoretical physics at the University of Florence and an associate member of the National Institute of Nuclear Physics and of the Institute for Complex Systems of the National Research Council. His research is focused on nonequilibrium statistical physics, and he has extensive experience teaching courses on statistical physics. He is the current president of the Italian Society of Statistical Physics.

Paolo Politi is a research director at the Institute for Complex Systems of the National Research Council and a fellow of the Marie Curie Association, the Alexander von Humboldt Foundation, and the Japan Society for the Promotion of Science. He currently teaches a course on stochastic processes and nonequilibrium statistical physics at the University of Florence. He was awarded the Outreach Prize of the Italian Physical Society.

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Nonequilibrium Statistical Physics

A Modern Perspective

Second Edition

Roberto Livi
University of Florence

Paolo Politi
Institute for Complex Systems of the National Research Council (CNR), Florence

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and
Jacques Villain (1934–2022)

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Preface to the Second Edition

The first edition of this book was published in 2017, and it was motivated by the desire to provide a textbook rather than a monograph. For this reason, its writing was led by the ambition to be as didactic as possible. On the other hand, the subject covered by its title, *Nonequilibrium Statistical Physics*, is potentially boundless: We made the choice to balance standard topics (Brownian motion, Langevin and Fokker–Planck equations, and linear response theory), more modern but inescapable subjects (transport processes and nonequilibrium phase transitions), and a personal selection of other topics (kinetic roughening, phase-ordering kinetics, and pattern formation).

Years after its publication, having discussed with many colleagues and having had the opportunity to use the book for our second-level master course, we thought that an enlarged and improved second edition might make sense from a scientific/pedagogical point of view. Fortunately, the first edition was successful enough to also make sense from an editorial point of view. We have therefore proposed a second edition whose main differences with the first edition are the addition of two new chapters and the heavy rewriting of four old chapters.

The new material is now contained in Chapters 1 and 3. The first chapter, "Kinetic Theory and the Boltzmann Equation" is completely new, while the third one, "Fluctuations and Their Probability," contains pedagogical introductions to large deviations and to stochastic thermodynamics. It also contains a discussion on generalized random walks, which is now framed in a more homogeneous context. Chapters 2 and 4, the core of standard topics of the first edition, are reissued with few changes.

The two old chapters on nonequilibrium phase transitions have instead been reorganized so as to distinguish between driven lattice models, Chapter 5, and absorbing phase transitions, Chapter 6. The next two chapters, Chapter 7 on kinetic roughening and Chapter 8 on phase ordering, have been rewritten to make them clearer and more fluid. Finally, Chapter 9 on pattern formation is almost unchanged.

Each chapter is accompanied by an opening, followed by an introductory section, and is closed by a final section with bibliographic notes. The reading of opening and introduction is recommended for those who aim at framing the great deal of methods and concepts contained in the book and specifically in each chapter. The final section lists some bibliographic advices for the specific chapter. A few suggested, general readings are: P. M. Chaikin and T. C. Lubensky, *Principles of Condensed Matter Physics* (Cambridge University Press, 2000); D. Chandler, *Introduction to Modern Statistical Mechanics* (Oxford University Press, 1987); P. L. Krapivsky, S. Redner, and E. Ben-Naim, *A Kinetic View of Statistical Physics* (Cambridge University Press, 2010); L. Peliti, *Statistical Mechanics* *in a Nutshell* (Princeton University Press, 2011); and J. P. Sethna, *Statistical Mechanics*: *Entropy, Order Parameters, and Complexity* (Oxford University Press, 2006).

The book is complemented by a dedicated website: https://sites.google.com/site/nespbook/. Readers are encouraged to report any errata or share their comments via the email address provided on the site.

Acknowledgments

Some colleagues have read and commented on parts of the book. So it is a great pleasure to thank Federico Corberi, Joachim Krug, and Alessandro Sarracino. Discussions with many colleagues helped us to clarify several questions. We are especially grateful to Franco Bagnoli, Filippo Colomo, Peter Grassberger, and Ruggero Vaia. Filippo Cherubini is acknowledged for having produced many figures of the book. PP thanks the Alexander von Humboldt Foundation for financial support, allowing him to spend a month at the Institute for Theoretical Physics of the University of Cologne to work on the present edition of this book.

Notations

Throughout this book, we assume the Boltzmann constant ($K_B = 1.380658 \times 10^{-23}\,\mathrm{J\,K^{-1}}$) to be equal to 1, which corresponds to measuring the temperature in joules or energy in kelvin.

We often use the symbol $N_A = 6.022141 \times 10^{23}$ to indicate the Avogadro number.

The space dimension is indicated by d.

As for the Fourier transform, we use the same symbol in the real and dual spaces, using the following conventions:

$$h(\mathbf{k}) = \int d\mathbf{x} \exp(-i\mathbf{k}\cdot\mathbf{x}) h(\mathbf{x}),$$
$$h(\mathbf{x}) = \frac{1}{(2\pi)^d} \int d\mathbf{k} \exp(i\mathbf{k}\cdot\mathbf{x}) h(\mathbf{k}).$$

Similarly, for the Laplace transform,

$$\omega(s) = \int_0^\infty dt e^{-st} \omega(t),$$
$$\omega(t) = \frac{1}{2\pi i} \mathrm{PV} \int_{a-i\infty}^{a+i\infty} ds e^{st} \omega(s).$$

Here, PV is the principal value of the integral and $a > a_c$, with a_c being the abscissa of convergence.

The friction coefficient of a particle of mass m, that is, the ratio between the force F_0 acting on it and its terminal drift velocity v_∞, is indicated by the symbol $\tilde{\gamma} = F_0/v_\infty$, but we frequently use the reduced friction coefficient, $\gamma = \tilde{\gamma}/m$, which has the dimension of the inverse of time. Similar reduced quantities are used in the context of Brownian motion.

The Helmholtz free energy is indicated by the symbol $F = U - TS$, and the free-energy density is indicated by f. We often use a free-energy functional, also called pseudo-free-energy functional or Lyapunov functional, and it is indicated by $\mathcal{F}$. It is the space integral of a function f, $\mathcal{F} = \int d\mathbf{x} f$. The susceptibility, indicated by χ, may be an extensive as well as an intensive quantity, depending on the context.

We use O and o to indicate the big and small O notations. For example, $\sin x = x + o(x)$ or $\sin x = x + O(x^3)$ for vanishing x.

When in the main text we cite a scientist, at the first occasion, we add their given name and nationality. If the family name of a scientist is only used to define an effect, an equation, or a model, we generally do not add further information.

Abbreviations

ASEP	asymmetric simple exclusion process
BD	ballistic deposition
BTW	Bak–Tang–Wiesenfeld
CA	cellular automaton
CDP	compact directed percolation
CH	Cahn–Hilliard
CIMA	chlorite-iodide-malonic-acid
CTRW	continuous time random walk
DK	Domany–Kinzel
dKPZ	deterministic Kardar–Parisi–Zhang
DLA	diffusion limited aggregation
DLG	driven lattice gas
DP	directed percolation
DyP	dynamical percolation
DW	domain wall
emf	electromotive force
EW	Edwards–Wilkinson
GL	Ginzburg–Landau
GOE	Gaussian orthogonal ensemble
GUE	Gaussian unitary ensemble
HD	high density
KLS	Katz–Lebowitz–Spohn
KMC	kinetic Monte Carlo
KPZ	Kardar–Parisi–Zhang
LD	low density
LG	lattice gas
LW	Lévy walk
MC	maximal current
MF	mean field
NESS	nonequilibrium steady state
PC	parity conserving
pdf	probability distribution function
PV	principal value
QFT	quantum field theory
RD	random deposition

RDR	random deposition with relaxation
RG	renormalization group
RSOS	restricted solid on solid
SH	Swift–Hohenberg
SOC	self-organized criticality
SS	single step
TASEP	totally asymmetric simple exclusion process
TDGL	time-dependent Ginzburg Landau
TW	Tracy–Widom
WV	Wolf–Villain