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Andries E. Brouwer , H. Van Maldeghem
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STRONGLY REGULAR GRAPHS

Strongly regular graphs lie at the intersection of statistical design, group theory, finite geometry, information and coding theory, and extremal combinatorics. This monograph collects all the major known results together for the first time in book form, creating an invaluable text that researchers in algebraic combinatorics and related areas will refer to for years to come. The book covers the theory of strongly regular graphs, polar graphs, rank 3 graphs associated to buildings and Fischer groups, cyclotomic graphs, two-weight codes and graphs related to combinatorial configurations such as Latin squares, quasi-symmetric designs and spherical designs. It gives the complete classification of rank 3 graphs, including some new constructions. More than 100 graphs are treated individually. Some unified and streamlined proofs are featured, along with original material including a new approach to the (affine) half spin graphs of rank 5 hyperbolic polar spaces.

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Preface

The present volume is a monograph on the topic of Strongly Regular Graphs. So far, no book-length treatment of this subject area has been available.

The topic of strongly regular graphs is an area where statistics, Euclidean geometry, group theory, finite geometry, and extremal combinatorics meet. The subject concerns beautifully regular structures, studied mostly using spectral methods, group theory, geometry and sometimes lattice theory.

Roughly around 1970–1980, Algebraic Combinatorics came up as a separate branch in mathematics. It turned out that the same structures were studied in statistics (for the design of experiments), in Euclidean geometry (e.g. in the construction of systems of equiangular lines), in group theory (where several sporadic groups arise as automorphism groups of a strongly regular graph), in coding theory (where association schemes provide a tool for obtaining bounds on the size of codes, and beautiful structures give rise to good and easy-to-decode codes), in the theory of special functions (where the spectral data of association schemes give rise to series of orthogonal polynomials), in finite geometry (where collinearity graphs of polar spaces are strongly regular), in extremal combinatorics, in cryptography, and elsewhere. More recently such very regular structures find some application in the theory of quantum computation (e.g. for mutually unbiased bases (MUBs) and symmetric, informationally complete, positive operator-valued measures (SICPOVMs)).

Axiomatizing the combinatorial information in the action of a finite permutation group G on a set X yields a hierarchy of combinatorial structures. A general group gives the structure of coherent configuration. For a transitive group one finds an association scheme. If the representation is multiplicity-free, the pair (G, K), where K is the point stabilizer in G, is called a Gelfand pair. The corresponding combinatorial object is a commutative association scheme. If G is generously transitive, one finds a symmetric association scheme. The simplest nontrivial case is that of a strongly regular graph, the combinatorial analog of a rank 3 group, where K has three orbits on $X \times X$.

Delsarte's 1973 thesis[1] defined the concept of (commutative) association scheme and showed the use of the linear programming bound. Bannai & Ito[2] introduced the term 'algebraic combinatorics', described as 'character-theoretical study of combinatorial objects', or 'group theory without groups'. Brouwer, Cohen & Neumaier[3] published a monograph on distance-regular graphs (that is, P- and Q-polynomial association schemes) of diameter at least 3 (where the strongly regular graphs are precisely the distance-regular graphs of diameter 2). They wrote 'Another book would be required to cover the present knowledge about strongly regular graphs (no such book is available at present)'. The present monograph fills this gap.

Various teams of authors, starting around 1980 with Van Lint and the present first author, contemplated writing such a book, but for various reasons such a project was never completed. Many years later J. I. Hall, at a 2011 meeting in Oisterwijk, again commented on the lack of a good source of information about strongly regular graphs more recent than Hubaut's 1975 survey,[4] and the project was rekindled.

This book was started with the aim to give the classification of rank 3 graphs and to describe these graphs, possibly as members of larger families, and give information such as parameters, group, cliques, cocliques, local structure, and characterization. Later, the project was widened to include the theory of general strongly regular graphs.

The bulk of the material is more or less well known. Many details are new. In particular, we give information about regular subsets that is often new. Our approach to the (affine) half spin graphs of rank 5 hyperbolic polar spaces is original and based on the idea of 'thickening' the Clebsch graph. We felt free to omit proofs that are rather technical, or that do not fit naturally into the line of development of the book.

Chapter 1 contains the fundaments. Chapters 2 and 3 find the finite polar geometries in a uniform way and describe the related graphs and substructures. Chapter 4 is a brief introduction to buildings,[5] and provides an explicit and elementary construction of the finite buildings of types E_6 and G_2. Chapter 5 is a very short introduction to the geometry related to the Fischer groups.[6] For later use, lax embeddings of symplectic copolar spaces are studied. Chapter 6 gives the main facts on the Golay codes and Witt designs, and contains a very short introduction to the Leech lattice.[7] Chapter 7 is about cyclotomy and difference sets, and the relation to two-weight codes. Chapter 8 contains combinatorial material that is partly new, with, for example, discussions of orthogonal arrays, quasi-symmetric designs, partial geometries, regular two-graphs,

1 Ph. Delsarte, *An algebraic approach to the association schemes of coding theory*, Philips Res. Rep. Suppl. **10** (1973).
2 E. Bannai & T. Ito, *Algebraic Combinatorics I*, Benjamin, 1984.
3 A. E. Brouwer, A. M. Cohen & A. Neumaier, *Distance-Regular Graphs*, Springer, 1989.
4 X. L. Hubaut, *Strongly regular graphs*, Discr. Math. **13** (1975) 357–381.
5 For a monograph, see P. Abramenko & K. S. Brown, *Buildings, Theory and Applications*, Springer, 2008.
6 For a monograph on the group theoretical side, see M. Aschbacher, *3-Transposition Groups*, Cambridge University Press, 1997.
7 For a monograph, see J. H. Conway & N. J. A. Sloane, *Sphere Packings, Lattices and Groups*, Springer, 1988.

spherical designs, randomness properties and much more. Chapter 9 discusses the p-rank of the adjacency matrix, in some cases a useful invariant that may distinguish graphs with the same parameters. The long Chapter 10 consists of a hundred sections discussing (more than) a hundred individual graphs in some more detail. In Chapter 11 we give the classification of rank 3 groups, and identify in each case the corresponding strongly regular graph. Everywhere there are extensive tables. Chapter 12 is just a table, listing all feasible parameter sets of strongly regular graphs with at most 512 vertices together with some information about existence and other details, with references to other parts of the book.

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