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978-1-316-51203-6 — Strongly Regular Graphs
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Excerpt
More Information

1

Graphs

This chapter collects some basic material on strongly regular graphs and gives some information about more general objects (distance-regular graphs and association schemes) that will be needed later.

1.1 Strongly regular graphs

A *graph* is a set X of *vertices* provided with a symmetric relation $\sim$ on X called *adjacency*, such that no $x \in X$ is adjacent to itself. If the graph is denoted by Γ, then its vertex set X is also denoted by $V\Gamma$. A pair of adjacent vertices is called an *edge*. If xy is an edge, then y is called a *neighbor* of x.

Let Γ be a finite graph. The *adjacency matrix* A of Γ is the square matrix indexed by the vertices of Γ such that $A_{xy} = 1$ when $x \sim y$, and $A_{xy} = 0$ otherwise. The *spectrum* of Γ is by definition the spectrum (eigenvalues and multiplicities) of A, considered as a real matrix. A nonzero (column) vector u, indexed by $V\Gamma$, is an eigenvector of A with eigenvalue θ when $Au = \theta u$, i.e., when $\sum_{y\sim x} u_y = \theta u_x$ for all x.

A graph Γ is *regular* of *degree* (or *valency*) k, for some integer k, when every vertex has precisely k neighbors.

Let Γ be finite with adjacency matrix A. The all-1 vector $\mathbf{1}$ (of appropriate length) is an eigenvector (with eigenvalue k) if and only if Γ is regular (of valency k). If Γ is regular of valency k, then the multiplicity of the eigenvalue k is the number of connected components of Γ. An eigenvalue θ of a regular graph is called *restricted* if it has an eigenvector orthogonal to $\mathbf{1}$.

A finite regular graph without restricted eigenvalues has at most one vertex. A finite regular graph with only one restricted eigenvalue is complete or edgeless. A *strongly regular graph* is a finite regular graph with precisely two restricted eigenvalues.

History

The term 'strongly regular graph' was first used by Bose [92]. An equivalent concept was studied by Bose & Shimamoto [97].

1.1.1 Parameters

Let Γ be a strongly regular graph, regular of valency k, with adjacency matrix A and restricted eigenvalues r, s, where $r > s$. Let J be the all-1 matrix of suitable size, so that $AJ = JA = kJ$. We have $(A - rI)(A - sI) = \mu J$ for some constant μ, so that $A^2 = \kappa I + \lambda A + \mu(J - I - A)$ for certain constants κ, λ, μ. Apparently $\kappa = k$ and $\lambda = \mu + r + s$ and $k - \mu = -rs$.

This can be stated in a combinatorial way: For $x, y \in V\Gamma$, the number of common neighbors of x, y is k when $x = y$, and λ when $x \sim y$, and μ when $x \nsim y$. One says that the strongly regular graph Γ has *parameters* (v, k, λ, μ), where $v = |V\Gamma|$ is the number of vertices. Conversely, if in a finite graph Γ, not complete and not edgeless, the number of common neighbors of two vertices is k, λ, μ depending on whether they are equal, adjacent or nonadjacent, then Γ is strongly regular, and the restricted eigenvalues r, s are found as the roots of $x^2 + (\mu - \lambda)x + (\mu - k) = 0$.

The combinatorial definition of k, λ, μ shows that these are nonnegative integers, and $0 \le \lambda \le k - 1$ and $0 \le \mu \le k$. By Perron-Frobenius' theorem, $k \ge r$. Since $\operatorname{tr} A = 0$ it follows that $s < 0$ and $r \ge 0$.

If $\mu \ne 0$, then the parameters are related by $v = 1 + k + k(k - 1 - \lambda)/\mu$.

From $(A - rI)(A - sI) = \mu J$ one gets the identity $(k - r)(k - s) = \mu v$.

History

The parameters $n, k, l, \lambda, \mu, r, s, f, g$ (with $n = v$ and $l = v - k - 1$) were perhaps first used in [419]. Earlier, Bose [92] used $v, n_1, n_2, p^1_{11}, p^2_{11}$.

1.1.2 Complement

If Γ is a strongly regular graph with parameters (v, k, λ, μ) and restricted eigenvalues r, s, then the complementary graph $\overline{\Gamma}$ (with the same vertex set as Γ, and where distinct vertices are adjacent if and only if they are nonadjacent in Γ) is also strongly regular, with parameters $(v, \bar{k}, \bar{\lambda}, \bar{\mu})$ and restricted eigenvalues $\bar{r}, \bar{s}$, where $\bar{k} = v - k - 1$, $\bar{\lambda} = v - 2k + \mu - 2$, $\bar{\mu} = v - 2k + \lambda$, $\bar{r} = -1 - s$, $\bar{s} = -1 - r$, as is immediately clear from the definitions and the fact that $\overline{\Gamma}$ has adjacency matrix $\overline{A} = J - I - A$.

1.1.3 Imprimitivity

A strongly regular graph Γ is called *imprimitive* when Γ or $\overline{\Gamma}$ is a nontrivial equivalence relation, equivalently, when $\lambda = k-1$ or $\mu = k$, equivalently, when $\mu = 0$ or $v = 2k-\lambda$, equivalently, when $s = -1$ or $r = 0$.

In the former case Γ is a disjoint union aK_m of a complete graphs of size m (and $v = am$, $k = m - 1$, $\lambda = m - 2$, $\mu = 0$, $r = m - 1$, $s = -1$), where $a > 1$.

In the latter case Γ is a complete multipartite graph $K_{a\times m}$ (and $v = am$, $k = (a - 1)m$, $\lambda = (a - 2)m$, $\mu = (a - 1)m$, $r = 0$, $s = -m$), again with $a > 1$.

(The graphs K_m and $K_{1\times m} = \overline{K_m}$ have only one restricted eigenvalue, namely -1 and 0 respectively, and hence are not strongly regular.)

For a primitive strongly regular graph it follows that $0 \le \lambda < k-1$ and $0 < \mu < k$ and $r > 0$ and $s < -1$. A primitive strongly regular graph is connected, and hence $k > r$.

The graph nK_2 is sometimes called a *ladder graph*. Its complement $\overline{nK_2} = K_{n\times 2}$ a *cocktail party graph*.

1.1.4 Spectrum

Let Γ be strongly regular, with spectrum k, r (with multiplicity f) and s (with multiplicity g). Then f, g can be solved from $1+f+g = v$ and $k+fr+gs = \mathrm{tr}\, A = 0$. The fact that f, g must be integers is a strong restriction on possible parameter sets.

If $f \neq g$, then one can also solve r, s from $r+s = \lambda - \mu$ and $fr + gs = -k$, and it follows that r, s are rational. Since they are also algebraic integers, they are integral in this case. On the other hand, if $f = g$, then $f = g = (v-1)/2$. Now $k = (\mu-\lambda)f = (\mu-\lambda)(v-1)/2$, and since $0 < k < v-1$ it follows that $k = (v-1)/2$ and $\mu = \lambda+1$. Now $v = 1+k+k(k-1-\lambda)/\mu$ yields $\mu = k-1-\lambda = k/2$, so that $(v, k, \lambda, \mu) = (4t+1, 2t, t-1, t)$ for a suitable integer t, and $r, s = (-1 \pm \sqrt{v})/2$. This is known as the ‘half case’. It occurs, e.g., for the Paley graphs (see §1.1.9). For further details, see §8.2.

Summary: if we are not in the half case, then the spectrum is integral.

Explicit expressions for f, g are $f = \frac{(s+1)k(k-s)}{\mu(s-r)}$ and $g = \frac{(r+1)k(k-r)}{\mu(r-s)}$.

The identity $\frac{vk(v-1-k)}{fg} = (r-s)^2$ (known as the *Frame quotient*, cf. [123] §2.2A, 2.7A) follows.

In particular, $v = (r-s)^2$ if and only if $\{f, g\} = \{k, v-k-1\}$.

1.1.5 Rank 3 permutation groups

A *permutation group* is a group G together with an action of G on some set X, that is, together with a map $G \times X \to X$ written $(g, x) \mapsto gx$, such that $1x = x$ and $g(hx) = (gh)x$ for all $g, h \in G$ and $x \in X$, where 1 is the identity element of G.

An *orbit* of G on X is a set of the form Gx for some $x \in X$. The G-orbits form a partition of X. The action (or the group) is called *transitive* when this partition has a single element only, that is, when $Gx = X$ for all $x \in X$. A set A is *preserved* by G when $gA = A$ for all $g \in G$.

The action of G on X induces an action of G on $X \times X$ via $g(x, y) = (gx, gy)$. If G is transitive, then it is said to be *of (permutation) rank r* when it has precisely r orbits on $X \times X$.

The action (or the group) is called *primitive* when there is no nontrivial equivalence relation $R \subseteq X \times X$ that is preserved by G. The trivial equivalence relations are the full set $X \times X$ and the diagonal $D = \{(x, x) \mid x \in X\}$.

Suppose G is a rank 3 permutation group on the set X. Then G has three orbits D, E, F on $X \times X$, where D is the diagonal. Now either E and F are inverse relations: $F = \{(y, x) \mid (x, y) \in E\}$, or E and F are symmetric. In the former case (X, E) is a complete directed graph, a tournament (and (X, F) is the opposite tournament). In the latter case (X, E) and (X, F) are a complementary pair of graphs. When X is finite, they are a complementary pair of strongly regular graphs: the group G acts as a group of automorphisms on the graphs (X, E) and (X, F), and since E and F are single orbits, G is transitive on ordered pairs of adjacent (nonadjacent) vertices, and the number of common neighbors of two vertices does not depend on the vertices chosen, but only on whether they are equal, adjacent or nonadjacent.

History

The study of rank 3 permutation groups was initiated by Higman [420].

1.1.6 Local graphs

If Γ is a graph, and x a vertex of Γ, then the *local graph* of Γ at x is the graph induced by Γ on the set of neighbors of x in Γ.

A graph Γ is called *locally* Δ (or *locally* X) where Δ is a graph and X a graph property, when all local graphs are isomorphic to Δ (or have property X).

For example, the icosahedron is the unique connected locally pentagon graph. Hall [391] determined all locally Δ graphs on at most 11 vertices, for all possible Δ, and determined for each graph Δ on at most 6 vertices whether there exists a locally Δ graph.

If Γ is a connected graph, and x a vertex of Γ, then the ith *subconstituent* of Γ (at x) is the graph induced on the set of vertices at (graph) distance i from x. If Γ is a strongly regular graph, and x a vertex of Γ, then the second subconstituent of Γ (at x) is the graph induced on the set of vertices other than x and nonadjacent to x.

1.1.7 Johnson graphs

Let Ω be a set, and $d \geq 0$ an integer. The *Johnson graph* $J(\Omega, d)$ is the graph that has as vertex set the set $\binom{\Omega}{d}$ of d-subsets of Ω, where two d-sets D, E are adjacent when $|D \cap E| = d - 1$. Suppose $|\Omega| \geq 2d$. Then $J(\Omega, d)$ has diameter d, and the symmetric group $\mathsf{Sym}(\Omega)$ acts as a group of automorphisms that is transitive of rank $d + 1$. If $|\Omega| = m$ one writes $J(m, d)$ instead of $J(\Omega, d)$.

The full group of automorphisms of $J(\Omega, d)$ is $\mathsf{Sym}(\Omega)$ when $|\Omega| > 2d > 0$, but $\mathsf{Sym}(\Omega) \times 2$ when $|\Omega| = 2d > 0$, and 1 when $d = 0$.

In particular, the graph $J(m, 2)$ (also called the *triangular graph* $T(m)$), where $m \geq 4$, is strongly regular. It has parameters $v = m(m-1)/2$, $k = 2(m-2)$, $\lambda = m - 2$, $\mu = 4$ and eigenvalues k, $r = m - 4$, $s = -2$ with multiplicities 1, $f = m - 1$, $g = m(m-3)/2$. The graph $T(m)$ is the line graph of the complete graph K_m on m vertices. The complement $\overline{T(5)}$ of $T(5)$ is the *Petersen graph* (§10.3).

These graphs are characterized by their parameters, except when $m = 8$. There are four graphs with the parameters $(v, k, \lambda, \mu) = (28, 12, 6, 4)$ of $T(8)$, namely $T(8)$ itself and three graphs known as the *Chang graphs* ([191, 192]), cf. §10.11.

1.1.8 Hamming graphs

Let Ω be a set, and $d \geq 0$ an integer. The *Hamming graph* $H(d, \Omega)$ is the graph that has as vertex set the set Ω^d of d-tuples of elements of Ω, where two d-tuples $(a_1, \ldots, a_d)$, $(b_1, \ldots, b_d)$ are adjacent when they have *Hamming distance* 1, i.e., when $a_i \neq b_i$ for a unique i. Suppose $|\Omega| \geq 2$. Then $H(d, \Omega)$ has diameter d, and its full group of automorphisms is the wreath product $\mathsf{Sym}(\Omega)$ wr $\mathsf{Sym}(d)$. This group is transitive of rank $d + 1$. If $|\Omega| = q$ one writes $H(d, q)$ instead of $H(d, \Omega)$.

In particular, the graph $H(2, q)$ (also called the *lattice graph* $L_2(q)$ or the $q \times q$ *grid*), where $q \geq 2$, is strongly regular. It has parameters $v = q^2$, $k = 2(q - 1)$, $\lambda = q - 2$, $\mu = 2$ and eigenvalues k, $r = q - 2$, $s = -2$ with multiplicities 1, $f = 2(q - 1)$, $g = (q - 1)^2$. The graph $H(2, q)$ is the line graph of the complete bipartite graph $K_{q,q}$. The graph $L_2(3)$ is isomorphic to its complement. It is the Paley graph (see §1.1.9) of order 9.

These graphs are characterized by their parameters, except when $q = 4$. There are two graphs with the parameters $(v, k, \lambda, \mu) = (16, 6, 2, 2)$, namely $L_2(4)$ and the *Shrikhande graph* ([649]), cf. §10.6.

The graph $H(d, q)$ is locally dK_{q-1}, the disjoint union of d complete graphs of size $q - 1$. The Shrikhande graph is locally a hexagon.

1.1.9 Paley graphs

Let $q = 4t + 1$ be a prime power. The *Paley graph* Paley(q) is the graph with the finite field $\mathbb{F}_q$ as vertex set, where two vertices are adjacent when they differ by a nonzero square. It is strongly regular with parameters $(4t + 1, 2t, t - 1, t)$. (The restriction $q \equiv 1 \pmod 4$ is to ensure that -1 is a square, so that the resulting graphs are undirected.)

Let $q = p^e$, where p is prime. The full group of automorphisms consists of the maps $x \mapsto ax^\sigma + b$ where $a, b \in \mathbb{F}_q$, a a nonzero square, and $\sigma = p^i$ with $0 \leq i < e$ ([186]). It has order $eq(q - 1)/2$.

Paley(5) is the pentagon. Paley(9) is the 3×3 grid. Paley(13) is a graph that is locally a hexagon. For a more detailed discussion, see §7.4.4.

1.1.10 Strongly regular graphs with smallest eigenvalue −2

A disjoint union of cliques has smallest eigenvalue $s = -1$. The pentagon has smallest eigenvalue $(-1 - \sqrt{5})/2$. All other strongly regular graphs satisfy $s \leq -2$. Seidel [642] determined the strongly regular graphs with smallest eigenvalue $s = -2$. There are three infinite families and seven more graphs:

(i) the complete n-partite graph $K_{n\times 2}$, with parameters $(v,k,\lambda,\mu) = (2n, 2n-2, 2n-4, 2n-2)$, $n \geq 2$,
(ii) the lattice graph $L_2(n)$, that is, the Hamming graph $H(2,n)$, that is, the $n \times n$ grid, with parameters $(v,k,\lambda,\mu) = (n^2, 2(n-1), n-2, 2)$, $n \geq 3$,
(iii) the triangular graph $T(n)$ with parameters $(v,k,\lambda,\mu) = (\binom{n}{2}, 2(n-2), n-2, 4)$, $n \geq 5$,
(iv) the Shrikhande graph (cf. §10.6), with parameters $(v,k,\lambda,\mu) = (16, 6, 2, 2)$,
(v) the three Chang graphs (cf. §10.11), with parameters $(v,k,\lambda,\mu) = (28, 12, 6, 4)$,
(vi) the Petersen graph (cf. §10.3), with parameters $(v,k,\lambda,\mu) = (10, 3, 0, 1)$,
(vii) the Clebsch graph (cf. §10.7), with parameters $(v,k,\lambda,\mu) = (16, 10, 6, 6)$,
(viii) the Schläfli graph (cf. §10.10), with parameters $(v,k,\lambda,\mu) = (27, 16, 10, 8)$.

More generally, the strongly regular graphs with fixed smallest eigenvalue are (i) complete multipartite graphs, (ii) Latin square graphs, (iii) block graphs of Steiner systems, (iv) finitely many further graphs, see Theorem 8.6.4.

We include a proof of Seidel's classification. (For different proofs, see [419] and [123], Theorem 3.12.4. See also below.)

Theorem 1.1.1 *A strongly regular graph with smallest eigenvalue -2 is one of the examples in (i)–(viii) above.*

Proof. We shall assume the classification of the graphs with the parameters of the examples. The proof here derives the possible parameters.

Let Γ be a strongly regular graph with parameters v,k,λ,μ and spectrum $k^1\ r^f\ s^g$, where $s = -2$. Then $\lambda = \mu + r - 2$ and $k = \mu + 2r$ (by §1.1.1), so that $k = 2\lambda - \mu + 4$.

If $\mu = 2$, then Γ has the parameters of $L_2(n)$ (for $n = r+2$), and hence is $L_2(n)$, or (if $n = 4$) the Shrikhande graph (cases (ii) and (iv)). If $\mu = 4$, then Γ has the parameters of $T(n)$ (for $n = r+4$), and hence is $T(n)$, or (if $n = 8$) a Chang graph (cases (iii) and (v)). Assume $\mu \neq 2, 4$.

From $1 + f + g = v$ and $k + fr - 2g = 0$ and $\mu v = (k-r)(k+2)$, we find $f = \frac{2v-k-2}{r+2} = \frac{(\mu+2r)(\mu+2r+2)}{\mu(r+2)}$.

Let an *m-claw* be an induced $K_{1,m}$ subgraph. Let a *quadrangle* be an induced C_4 subgraph. Let $x \sim a, b$ with $a \nsim b$. If $\{x, a, b\}$ is contained in c 3-claws and in q quadrangles, then $k = 2 + 2\lambda - (\mu - 1 - q) + c$ so that $c + q = 1$.

First consider the case where the graph contains a 3-claw. Let $x \sim a, b, c$ with mutually nonadjacent a, b, c. We shall show that $v = 2k + 4$ and Γ is one of the examples (iv)–(vi).

For a list of vertices Z, let $N(Z)$ ('near') be the set of vertices adjacent to each z in Z, and $F(Z)$ ('far') the set of vertices not in Z and nonadjacent to each z in Z. Since the $k-\lambda-1 = r+1$ vertices in $N(x) \cap F(a)$ are in $\{b,c\} \cup N(b,c) \setminus \{x\}$, we have $r \leq \mu$. Since the $k - \lambda$ vertices in $(N(a) \cap F(x)) \cup \{a\}$ are among the $\overline{\lambda} = v - 2k + \mu - 2$ vertices of $F(b,c)$, we have $v \geq 5r + \mu + 4$. Since $\mu v = (k-r)(k+2)$ we have $v = 3r + \mu + 2 + \frac{2r(r+1)}{\mu}$ so that $\mu \leq r$. It follows that $\mu = r$, $\lambda = 2r-2$, $k = 3r$,

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$v = 6r+4 = 2k+4$, $f = 9 - \frac{12}{r+2}$ so that $r \in \{1,2,4,10\}$. For $r = 1,2,4$ we are in case (vi), (iv), (v), respectively. The case $(v,k,\lambda,\mu) = (64,30,18,10)$ has $f = 8$, which violates the absolute bound $v \le \frac{1}{2}f(f+3)$ (Proposition 1.3.14 below).

Now assume that Γ does not contain 3-claws. Since $c+q = 1$, each 2-claw is in a unique quadrangle. It follows that μ is even, say $\mu = 2m$, and if $a \nsim b$, then $N(a,b)$ induces a $K_{m\times 2}$. If moreover $d \sim a$, $d \nsim b$, then d is adjacent to precisely m vertices of $N(a,b)$. (If $x, y \in N(a,b)$ with $x \nsim y$, then d cannot be nonadjacent to both x and y, since $(a;x,y,d)$ would be a 3-claw, and d cannot be adjacent to both x and y, since we already see the μ common neighbors of x and y in $N(a,b) \cup \{a,b\}$.)

Let b be a vertex, and consider the graph induced on $F(b)$. It is strongly regular or complete or edgeless with parameters $(v_0,k_0,\lambda_0,\mu_0) = (v-k-1, k-\mu, \lambda-m, \mu)$. If it is edgeless, then $k = \mu$, so that Γ is imprimitive, and we are in case (i). If it is complete, then $v-k-1 = k-\mu+1$ so that $(\mu+2r)(r+1) = \mu(2r+1)$, hence $\mu = 2(r+1)$ and $f = 8 - \frac{12}{r+2}$, so that $r \in \{1,2,4,10\}$. For $r = 1$ we have $T(5)$ (in case (iii)), for $r = 2$ the Clebsch graph (case (vii)), and $r = 4$ ($v = 28$, $f = 6$) and $r = 10$ ($v = 64$, $f = 7$) both violate the absolute bound.

So we may assume that $F(b)$ induces a strongly regular graph Δ. Since $k_0 = 2\lambda_0 - \mu_0 + 4$, also Δ has smallest eigenvalue -2, and the other restricted eigenvalue is $r_0 = r - m$ with multiplicity $f_0 = \frac{2r(r+1)}{m(r-m+2)}$. By induction we already know Δ (and it does not contain 3-claws) so either $\mu \in \{6,8\}$, or Δ is $K_{n\times 2}$. For $\mu = 6$ there are no feasible parameters. For $\mu = 8$ we find the Schläfli graph (case (viii)). If Δ is $K_{n\times 2}$, then $(v,k,\lambda,\mu) = (6n-3, 4n-4, 3n-5, 2n-2)$, $r = n-1$, $f = 8 - \frac{12}{r+2}$, so that $r \in \{1,2,4,10\}$. For $r = 1$ we have $L_2(3)$ (in case (ii)), for $r = 2$ we have $T(6)$ (in case (iii)), for $r = 4$ the Schläfli graph (case (viii)), and $r = 10$ ($v = 63$, $f = 7$) violates the absolute bound. □

Root systems

In fact it is possible to find all graphs with smallest eigenvalue ≥ -2. By the beautiful theorem of Cameron, Goethals, Seidel & Shult [179] (see also [123], §3.12 and [132], §8.4) such a graph is either a generalized line graph or is one in a finite (but large) collection.

(Sketch of the proof: Consider $A + 2I$. It is positive semidefinite, so one can write $A + 2I = M^\top M$. Now the columns of M are vectors of squared length 2 with integral inner products, and this set of vectors can be completed to a root system. By the classification of root systems one gets one of A_n, D_n, E_6, E_7 or E_8. In the first two cases the graph was a generalized line graph. In the latter three cases the graph is finite: at most 36 vertices, each vertex of degree at most 28. If the graph was regular, it has at most 28 vertices, and each vertex has degree at most 16. For details, see [123], Theorem 3.12.2, or [132], Chapter 8.)

There is a lot of literature describing manageable parts of this large collection, and related problems. A book-length treatment is Cvetković et al. [249].

1.1.11 Seidel switching

Instead of the ordinary adjacency matrix A, Seidel considered the *Seidel matrix* S of a graph, with zero diagonal, where $S_{xy} = -1$ if $x \sim y$, and $S_{xy} = 1$ otherwise. These matrices are related by $S = J - I - 2A$.

Let Γ be a graph with vertex set X. Let $Y \subseteq X$. The graph Γ' obtained by *switching* Γ with respect to Y is the graph with vertex set X, where two vertices that are both inside or both outside Y are adjacent in Γ' when they are adjacent in Γ, while a vertex inside Y is adjacent in Γ' to a vertex outside Y when they are not adjacent in Γ. If Γ has Seidel matrix S, then Γ' has Seidel matrix S' where S' is obtained from S by multiplying each row and each column with index in Y by -1. It follows that S and S' have the same spectrum.

If Γ' is obtained from Γ by switching w.r.t. Y, and Γ'' is obtained from Γ' by switching w.r.t. Z, then Γ'' is obtained from Γ by switching w.r.t. $Y \triangle Z$. It follows that graphs related by switching fall into equivalence classes (called *switching classes*). Two graphs in the same switching class are called *switching equivalent*.

If two regular graphs of the same valency are switching equivalent, then they have the same ordinary spectrum. This happens precisely when each vertex inside (outside) the switching set is adjacent to half of the vertices outside (inside, respectively) the switching set. For example, the Shrikhande graph is obtained from the 4×4 grid by switching w.r.t. a diagonal.

It may happen that two strongly regular graphs of different valencies are switching equivalent. If that happens, then they are related to regular 2-graphs (see §1.1.12).

Proposition 1.1.2 *Let Γ be a strongly regular graph with parameters (v, k, λ, μ) and spectrum $k^1\ r^f\ \ s^g$. Let Δ be a strongly regular graph of valency $\ell > k$ switching equivalent to Γ. Then (i) Δ has spectrum $\ell^1\ r^{f-1}\ s^{g+1}$, (ii) $\frac{1}{2}v = k - s = \ell - r$, (iii) $k - r = 2\mu$, (iv) $\frac{1}{2}v = 2k - \lambda - \mu$, (v) any switching set from Γ to Δ has size $\frac{1}{2}v$ and is regular of degree $k - \mu$.*

Proof. (i)–(iv) The Seidel matrices $S = J - I - 2A$ of Γ and Δ have the same spectrum $(v-1-2k)^1\ (-1-2r)^f\ \ (-1-2s)^g$, and if $k < l$ it follows that $v - 1 - 2k = -1 - 2s$ and $v - 1 - 2\ell = -1 - 2r$. Since $(k - r)(k - s) = \mu v$ for all strongly regular graphs, it follows from $k - s = \frac{1}{2}v$ that $k - r = 2\mu$. Since $r + s = \lambda - \mu$ for all strongly regular graphs, we find $\frac{1}{2}v = k - s = k - r - s + r = k + \mu - \lambda + k - 2\mu = 2k - \lambda - \mu$.

(v) Suppose Δ is obtained from Γ by switching w.r.t. a set U of size u. Let $x \in U$ have k_1 neighbors in U and k_2 outside. Then $k = k_1 + k_2$ and $\ell = k_1 + v - u - k_2$, so that k_1 and k_2 can be expressed in terms of k, ℓ, u, v and are independent of x. Similarly, if $y \notin U$ has k_3 neighbors in U and k_4 outside, then $k = k_3 + k_4$ and $\ell = u - k_3 + k_4$, so that k_3 and k_4 are independent of y. Counting the number of edges with one end in U in two ways, we find $k_2 u = k_3(v - u)$, and since $k_2 = \frac{1}{2}(k - \ell - u + v)$ and $k_3 = \frac{1}{2}(k - \ell + u)$ this simplifies to $(k - \ell)u = (k - \ell)(v - u)$, so that $u = \frac{1}{2}v$, $k_2 = k_3$, $k_1 = k_4$. □

The Seidel matrix plays a role in the description of regular two-graphs and of sets of equiangular lines, cf. [132], Chapter 10. The condition $\frac{1}{2}v = 2k - \lambda - \mu$ is necessary and sufficient for a strongly regular graph to be associated to a regular two-graph, cf. [132], 10.3.2(i), and see below.

History

The Seidel matrix was introduced in Seidel [641].

1.1.12 Regular two-graphs

A *two-graph* $\Omega = (V, \Delta)$ is a finite set V provided with a collection Δ of unordered triples from V, such that every 4-subset of V contains an even number of triples from Δ. The triples from Δ are called *coherent*.

From a graph $\Gamma = (V, E)$, one can construct a two-graph $\Omega = (V, \Delta)$ by calling a triple from V coherent if the three vertices induce a subgraph in Γ with an odd number of edges. One checks that Ω is a two-graph. It is called the two-graph associated to Γ. Switching equivalent graphs have the same associated two-graph.

Conversely, from any two-graph $\Omega = (V, \Delta)$, and any fixed $w \in V$, we can construct a graph $\Gamma = \Omega_w$ with vertex set V as follows: let w be an isolated vertex in Γ, and let any two other vertices x, y be adjacent in Γ if $\{w, x, y\} \in \Delta$. Then Ω is the two-graph associated to Γ.

Thus we have established a one-to-one correspondence between two-graphs and switching classes of graphs.

Let $\Omega = (V, \Delta)$ be a two-graph, and $w \in V$. The *descendant* of Ω at w is the graph Ω^*_w, obtained from Ω_w by deleting the isolated vertex w.

A two-graph (V, Δ) is called *regular* (of degree a) if every unordered pair from V is contained in exactly a triples from Δ. The two-graph $\Omega = (V, \Delta)$ with $v = |V|$ vertices and $0 < |\Delta| < \binom{v}{3}$ is regular if and only if any descendant is strongly regular with parameters $(v-1, k, \lambda, \mu)$ where $\mu = k/2$ (and then this holds for all descendants). If this is the case, then $a = k$ and $v = 3k - 2\lambda$.

See also §8.10 and [132], §10.3.

History

Regular two-graphs were introduced by G. Higman. See also Taylor [677].

1.1.13 Regular partitions and regular sets

Let Γ be a finite graph with vertex set X. A partition $\{X_1, \ldots, X_m\}$ of X is called *regular* or *equitable* when there are numbers e_{ij}, $1 \le i, j \le m$, such that each vertex of X_i is adjacent to precisely e_{ij} vertices in X_j. In this situation the matrix $E = (e_{ij})$ of order m is called the *quotient matrix* of the partition.

If θ is an eigenvalue of E, say $Eu = \theta u$, then θ is also an eigenvalue of Γ, for the eigenvector that is constant u_i on X_i. And conversely, the eigenvalues of Γ that belong to eigenvectors constant on each X_i are eigenvalues of E.

Let Γ be finite and regular of valency k. A subset Y of the vertex set X is called *regular* (of *degree d* and *nexus e*) when the partition $\{Y, X \setminus Y\}$ is regular (and $e_{11} = d$, $e_{21} = e$ where $X_1 = Y$). Now the quotient matrix $E = \left(\begin{smallmatrix} d & k-d \\ e & k-e \end{smallmatrix}\right)$ has eigenvalues k and $d - e$, so that $d - e$ is an (integral) eigenvalue of Γ.

A regular set is also called an *intriguing set* ([263]).

Proposition 1.1.3 *Let Γ be strongly regular with parameters (v, k, λ, μ). If Y and Y' are regular sets of degrees d, d' and nexus e, e' belonging to different eigenvalues $d - e$ and $d' - e'$ other than k, then $|Y \cap Y'| = ee'/\mu$.*

Proof. The vector u that is 1 on Y and $a := \frac{-e}{k-d}$ outside Y is an eigenvector of the adjacency matrix A of Γ with eigenvalue $\theta := d - e$. Here $a \neq 1$ since $\theta \neq k$. The characteristic vector of Y is $\chi_Y = \frac{1}{1-a}u - \frac{a}{1-a}\mathbf{1}$, where $\frac{a}{1-a} = \frac{-e}{k-\theta}$. Similarly for Y'. Since $u, u', \mathbf{1}$ are mutually orthogonal, $(\mathbf{1}, \mathbf{1}) = v$, and $\mu v = (k - \theta)(k - \theta')$, we have $|Y \cap Y'| = (\chi_Y, \chi_{Y'}) = \frac{ee'}{(k-\theta)(k-\theta')} v = ee'/\mu$. □

We also see that $|Y| = (\chi_Y, \mathbf{1}) = \frac{ev}{k-\theta}$ with $\theta = d - e$.

The collection of regular sets belonging to the same eigenvalue $\theta = d - e$ (together with $\emptyset$ and X) is closed under taking complements, under taking disjoint unions, and under removal of one set from one containing it.

In descendants of regular two-graphs, switching sets are regular sets.

Proposition 1.1.4 *Let Γ be strongly regular with parameters (v, k, λ, μ) and restricted eigenvalues r, s, where $k = 2\mu$. Let Y be a regular set in Γ of degree d and nexus e. If $|Y| = k - c$, where $\{c, d - e\} = \{r, s\}$, then adding an isolated vertex and switching w.r.t. Y yields a strongly regular graph with parameters $(v + 1, k - c, \lambda - c, \mu - c)$.* □

1.1.14 Inequalities for subgraphs

We give inequalities that must hold for a graph Γ to have certain induced subgraphs. Additional regularity holds when there are such subgraphs and the inequality holds with equality.

Interlacing

Let Γ be a finite graph with adjacency matrix A, and let $\Pi = \{X_1, \ldots, X_m\}$ be a partition of a subset of $V\Gamma$. The *quotient matrix* of A w.r.t. Π is the matrix B of order m where B_{ij} is the average row sum of the submatrix $A(i, j)$ of A that has rows indexed by X_i and columns indexed by X_j. If each $A(i, j)$ has constant row sums, and Π partitions $V\Gamma$, then Π is an equitable partition of Γ, and B is a quotient matrix in the sense of §1.1.13 (hence the present definition generalizes the previous one).