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Excerpt
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Part I

Special Relativity

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The Geometry of Special Relativity

1.1 Introduction

1.1.1 Classical Physical Systems

A classical[1] physical system consists of three parts:

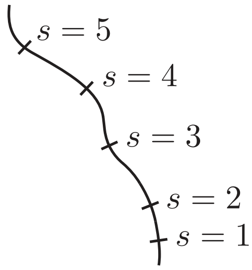
1. **Four-dimensional spacetime:** the *arena* of classical physics. We label a point in spacetime (an "event") by its coordinates:

$$x^\mu = (x^0, x^i) = (ct, \mathbf{x}), \tag{1.1}$$

   where x^0 represents the time (we'll use units such that $c = 1$)[2] and $\mathbf{x}$ the position. Greek indices near the middle of the alphabet $(\lambda, \mu, \nu, \ldots)$ run from 0 to 3; Roman indices near the middle $(i, j, k, \ldots)$ run from 1 to 3.
2. **Particles and fields:** the *entities* of classical physics.

   (a) **Particles:** A particle is a structureless point object. Its *location*, $\mathbf{x}(t)$, as a function of time, tells you everything there is to say about it (beyond fixed properties such as mass and charge).[3] In 4-vector notation we represent the particle's trajectory (its **world line**) by $x^\mu(s)$, where s is any parameter used to denote points along the curve ($f(s)$ would do just as well, for any monotonic function f):

[1] In this book "classical" means "pre-quantum"; it *includes* special relativity.
[2] It's easy to reinsert the c's, when necessary, by dimensional analysis.
[3] We could treat point objects with spin, but let's keep things simple; in this course "particle" means spin 0.



(b) **Fields:** A field is a function of position and time:

$$\varphi^{\alpha}(x). \tag{1.2}$$

Here α labels the components: one of them, if the field is temperature; six of them, in the case of electromagnetism. (In expressions like this x stands for the four components of x^{μ}.)

3. **Dynamics:** the *laws of motion*.

1.1.2 Symmetries

A **symmetry** is an operation that leaves an object or a system unchanged (**invariant**). A square, for example, is invariant under rotations (about a perpendicular axis through its center) by 90°, or 180°, or 270°, or reflections (in either diagonal, or a bisector of two opposite sides). Of particular importance to us are invariances of the laws of motion,[4] transformations that carry one possible motion into another. We stipulate that an invariance must have a well-defined inverse.[5]

Mathematically, the invariances of a system form a **group**.

Definition: A group, G, is a set of elements $(a, b, c, \ldots)$ and a law of "multiplication," with the following properties:

1. It is **closed**: if a and b are in G, so is their product, ab.
2. It is **associative**: $a(bc) = (ab)c$.
3. It contains a (unique) unit element, 1, such that $1a = a1 = a$ for every element a.
4. Each element a has a (unique) inverse, a^{-1}, such that $a^{-1}a = aa^{-1} = 1$.

[4] The ancient Greeks thought symmetries pertain to the actual motion: celestial objects ought to move on circles, because a circle is the most perfect (symmetrical) shape. But since the time of Newton we have understood that the more significant invariances apply to the *equations of motion*, and hence to the collection of all *possible* motions—the set of *solutions* to the equations of motion. The sun's gravitational field is spherically symmetric, but planetary orbits don't directly exhibit that symmetry—they're *elliptical*.

[5] This restriction eliminates trivial possibilities such as mapping all points on a particle trajectory onto some fixed point (sitting still at one point being—usually—a solution to the equations of motion). It is necessary in order to ensure that the invariances form a group.

For example, the real numbers (except 0), with multiplication defined in the usual way, constitute an **Abelian** (commutative: $ab = ba$) group. Another group is the set of permutations of three objects (this group is *not* Abelian). We are interested here in the group of invariances of classical physics; "multiplication" in this context means application of two transformations in succession.

Example 1.1

Imagine a quantum mechanical system with nondegenerate energy levels. The state of the system at time $t = 0$ can be expanded in terms of the energy eigenstates:

$$|\psi(0)\rangle = \sum a_n \, |n\rangle, \tag{1.3}$$

and at any later time

$$|\psi(t)\rangle = \sum a_n e^{-iE_n t/\hbar} \, |n\rangle. \tag{1.4}$$

But the *phase* of $|n\rangle$ is arbitrary; physical predictions are unaffected by the transformation

$$|n\rangle \to e^{i\theta_n} |n\rangle, \tag{1.5}$$

for any collection of real numbers θ_n (independent of position and time). This is a huge invariance group, with an infinite number of parameters (if there are infinitely many eigenstates). But for the most part it is a *useless* invariance, which does not help us to solve the equations of motion.

So there exist trivial, accidental, or otherwise inconsequential invariances. One particularly *useful* class consists of the *geometrical* invariances of space and time: translations, rotations, dilations[6] (stretching), and so on. *Question:* What is the group of geometrical invariances of classical physics—the geometrical transformations that leave the laws of classical physics unchanged? A geometrical transformation is a change of coordinates:

$$x^\mu \to x'^\mu = y^\mu(x). \tag{1.6}$$

In the case of a particle trajectory,

$$x^\mu(s) \to y^\mu(x(s)). \tag{1.7}$$

Fields are more complicated, because not only do the *components* mix (if it's a vector field, and we're performing a rotation, the $\hat{\mathbf{x}}$ component will pick up $\hat{\mathbf{y}}$ and $\hat{\mathbf{z}}$ terms), but the *argument* (x) must be expressed in terms of the new coordinates (y): schematically,

[6] Eds. Coleman calls them "dilatations." Presumably permute:dilate::permutation:dilatation. But most modern authors use "dilation," and "dilatation" seems unnecessarily awkward.

$$\varphi^\alpha(x) \to [\varphi^\alpha(x)]' = F[\varphi^\beta(y^{-1}(x))], \tag{1.8}$$

where F is some function denoting the transformation (mixing) of the components (φ^β), and y^{-1} is the inverse of Eq. 1.6. In words, the new fields at point y are some functions of the old fields at the point x that got mapped into y.

1.2 Poincaré Invariance

1.2.1 Geometrical Symmetries of Classical Physics

We'll focus for the moment on the case of particles. If there were *no* laws of motion (i.e. if *every* particle motion were possible), then *any* geometrical transformation would be an invariance. We'll whittle down this (huge) group by invoking some *actual* laws of motion:

1. **Newton's first law.** The allowed motions for a *free* particle are straight lines in spacetime, so the invariance group must (at a minimum) *take straight lines into straight lines*. One way to characterize a straight line is

$$x^\mu(s) = v^\mu s + b^\mu, \text{ where } v^\mu = \frac{dx^\mu}{ds} \text{ and } b^\mu \text{ are constants,} \tag{1.9}$$

which is the general solution to the differential equation

$$\frac{d^2x^\mu}{ds^2} = 0. \tag{1.10}$$

But wait: we *could* have used a different parameterization (say, s^3 instead of s); then

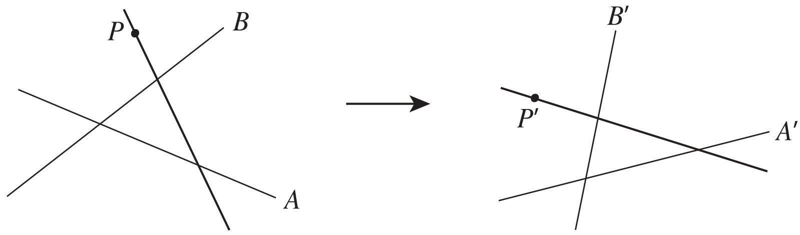
$$x^\mu(s) = v^\mu s^3 + b^\mu. \tag{1.11}$$

So Eq. 1.10 is not a reliable way to characterize straight lines—it's *sufficient*, but not *necessary*. Maybe a straight line satisfying 1.10 is transformed into a straight line that *doesn't* satisfy 1.10. In point of fact this worry is misguided: an invariance that carries straight lines into straight lines *automatically* takes linearly parameterized straight lines 1.9 into linearly parameterized straight lines.

Proof: For transformations that carry straight lines into straight lines:

(a) **Intersecting (or nonintersecting) straight lines go into intersecting (nonintersecting) straight lines.** If intersecting lines transformed into nonintersecting lines, the transformation for the point of intersection would be ill defined, since it would have to go to two different points—one on each line. And because we have stipulated that invariances have well-defined inverses, the same goes for nonintersecting to intersecting.

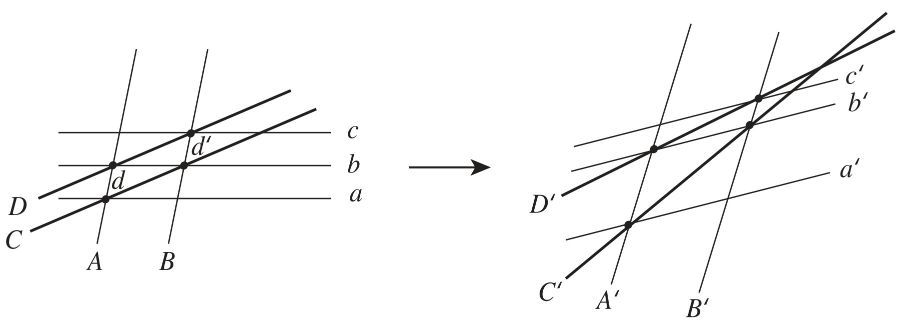
(b) **Planes go into planes.** Let P be a point in the plane defined by intersecting lines A and B (but not *on* either line), and draw a line from P intersecting A and B:



This line transforms into a line intersecting A' and B', so P' lies in the plane defined by A' and B'.

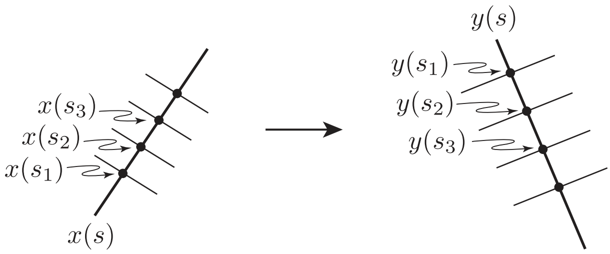
(c) **Parallel lines go into parallel lines.** This follows from (a) and (b).

(d) **Equidistant coplanar parallel lines go into equidistant coplanar parallel lines.** We know that coplanar parallel lines go into coplanar parallel lines, but could it be that equidistant ones (a, b, c) go into *non*equidistant ones (a', b', c')?



No: draw line A, and let the distance between its intersections with a and b be d. Now draw line B, parallel to A, and construct lines C and D, passing through the four intersections. By simple geometry, C and D are parallel (because a, b, and c are equidistant), and $d' = d$. However, unless a', b', and c' are *also* equidistant, C' and D' will *not* be parallel, violating (c).

So the transformation $x(s) \to y(x(s))$ takes equal intervals $(x(s_3) - x(s_2) = x(s_2) - x(s_1))$ into equal intervals $(y(s_3) - y(s_2) = y(s_2) - y(s_1))$, preserving the linear parameterization. QED



Under the transformation 1.6,

$$x^\mu \to y^\mu(x^\nu),$$

derivatives transform (by the chain rule)[7] as

$$\frac{dx^\mu}{ds} \to \frac{dy^\mu}{ds} = \frac{\partial y^\mu}{\partial x^\nu}\frac{dx^\nu}{ds}, \tag{1.12}$$

$$\frac{d^2x^\mu}{ds^2} \to \frac{d^2y^\mu}{ds^2} = \frac{\partial y^\mu}{\partial x^\nu}\frac{d^2x^\nu}{ds^2} + \frac{\partial^2 y^\mu}{\partial x^\nu \partial x^\lambda}\frac{dx^\nu}{ds}\frac{dx^\lambda}{ds}. \tag{1.13}$$

Because all straight lines ($d^2x^\mu/ds^2 = 0$) must transform into straight lines ($d^2y^\mu/ds^2 = 0$), it follows that invariances consistent with Newton's first law satisfy

$$\frac{\partial^2 y^\mu}{\partial x^\nu\, \partial x^\lambda} = 0 \tag{1.14}$$

(for all μ, ν, λ). The general solution is a *linear* function of x:

$$y^\mu = M^\mu_{\ \nu}\, x^\nu + b^\mu, \tag{1.15}$$

where the 16 elements of $M^\mu_{\ \nu}$ and the 4 components of b^μ are constants. (As a 4×4 matrix, $\det M \neq 0$, since $y(x)$ must have an inverse.) Newton's first law has reduced the geometrical invariances to a 20-parameter group, the **inhomogeneous affine group** (in four dimensions); with $b^\mu = 0$ it becomes the **homogeneous affine group**.

2. **Constancy of the velocity of light.** In empty space, light travels in straight lines, and according to special relativity the speed of light (in vacuum) is a universal constant, independent of the velocity of the source or the observer. If a light signal travels from point $\mathbf{x}$ to point $\mathbf{x}'$, departing at time t and arriving at time t', then

[7] We use the **Einstein summation convention**, whereby repeated indices are summed. Thus the third term in Eq. 1.12 carries an implicit summation sign, $\sum_{\nu=0}^{3}$.

$$c(t' - t) = |\mathbf{x}' - \mathbf{x}|, \tag{1.16}$$

or (setting $c = 1$)

$$(t' - t)^2 = (\mathbf{x}' - \mathbf{x})^2 = \sum_{i=1}^{3} [(x^i)' - x^i]^2. \tag{1.17}$$

Introducing the **metric tensor**[8]

$$g_{\mu\nu} \equiv \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & -1 \end{pmatrix}, \tag{1.18}$$

we have

$$(x' - x)^\mu g_{\mu\nu} (x' - x)^\nu = 0. \tag{1.19}$$

The constancy of the speed of light means that if x and x' satisfy Eq. 1.19, then so too must the transformed coordinates y and y'. What does this tell us about M and b? Well,

$$x^\mu \to y^\mu = M^\mu_{\ \nu} x^\nu + a^\mu \;\Rightarrow\; (y' - y)^\mu = M^\mu_{\ \nu} (x' - x)^\nu, \tag{1.20}$$

so

$$M^\mu_{\ \kappa} (x' - x)^\kappa g_{\mu\nu} M^\nu_{\ \sigma} (x' - x)^\sigma = 0, \tag{1.21}$$

or

$$(x' - x)^\kappa \left[M^\mu_{\ \kappa}\, g_{\mu\nu}\, M^\nu_{\ \sigma} \right] (x' - x)^\sigma = 0. \tag{1.22}$$

This must hold for any x and x' satisfying Eq. 1.19. It follows that[9]

$$M^\mu_{\ \kappa}\, g_{\mu\nu}\, M^\nu_{\ \sigma} = \lambda g_{\kappa\sigma} \tag{1.23}$$

for some constant λ; or, in matrix notation,[10]

$$M^T g M = \lambda g. \tag{1.24}$$

[8] Some authors use the other **signature** $(-, +, +, +)$; it doesn't matter, as long as you are consistent.

[9] Although 1.23 obviously *guarantees* 1.22, it is not so clear that it is *required* by 1.22. But remember that this must hold for *any* x and x' satisfying 1.19, and from this it is not hard to show that 1.23 is in fact *necessary*.

[10] Reading left to right, the first index (whether up or down) is the *row*, and the second (up or down) is the column. The significance of upness and downness will appear in due course. The superscript T denotes the transpose: $(M^T)^\mu_{\ \kappa} = M_\kappa^{\ \mu}$.

What sorts of transformations remain, after invoking Newton's first law and the constancy of the speed of light? We can factor the matrix M as follows:

$$M = M_1 M_2, \text{ where } M_1 = |\det M|^{1/4} I \text{ and } M_2 = \frac{M}{|\det M|^{1/4}} \tag{1.25}$$

(I is the unit matrix). Thus any M is the product of a pure **dilation** M_1,

$$M_1 = \alpha I, \text{ so } M_1^T g M_1 = \alpha^2 g \text{ and hence } \lambda_1 = \alpha^2, \tag{1.26}$$

and a dilation-free term M_2 with determinant ± 1, for which

$$(\det M_2)(\det g)(\det M_2) = \lambda_2^4 (\det g) \;\Rightarrow\; \lambda_2^4 = 1 \;\Rightarrow \lambda_2 = \pm 1. \tag{1.27}$$

Actually, the negative sign is impossible,[11] so (in view of Eq. 1.26) $\lambda = \lambda_1 \lambda_2$ is in fact always positive.

3. **Eliminating dilations.** *Question:* Is our universe invariant under dilations? Imagine performing the Cavendish experiment to measure the gravitational force between two point masses:

$$F = G\frac{m_1 m_2}{r^2}, \tag{1.28}$$

giving an acceleration to m_1 in the amount

$$a_1 = G\frac{m_2}{r^2}. \tag{1.29}$$

Under a dilation (change of scale),

$$r \to \lambda r, \quad t \to \lambda t, \quad a \to \lambda^{-1} a. \tag{1.30}$$

So if dilation doesn't affect G or m_2, then a_1 goes like λ^{-1} but Gm_2/r^2 goes like λ^{-2}. Since G is a universal constant, it can't depend on λ, and since there is no mass continuum (no electron, for example, with slightly larger or smaller mass), mass cannot depend continuously on λ. *Conclusion:* Our universe is *not* invariant under dilations.[12]

[11] This follows from **Sylvester's law of inertia**; see, for instance, S. MacLane and G. Birkhoff, *A Survey of Modern Algebra*, 3rd ed., Macmillan (1965) p. 254. In essence, if $M^T g M = -g$ then $Q \equiv (x^0)^2 - (x^1)^2 - (x^2)^2 - (x^3)^2 = -(y^0)^2 + (y^1)^2 + (y^2)^2 + (y^3)^2$, so there is a 3-dimensional subspace ($x^0 = 0$) in which $Q < 0$, and another 3-dimensional subspace ($y^0 = 0$) in which $Q > 0$. But the entire space has only four dimensions, so this is impossible.

[12] This still leaves open the possibility of invariance under *combined* dilations and Lorentz transformations. We'll eliminate that option in Section 1.2.6.