

HANDBOOK OF CONSTRUCTIVE MATHEMATICS

Constructive mathematics – mathematics in which ‘there exists’ always means ‘we can construct’ – is enjoying a renaissance. Fifty years on from Bishop’s groundbreaking account of constructive analysis, constructive mathematics has spread out to touch almost all areas of mathematics and to have profound influence in theoretical computer science. This handbook gives the most complete overview of modern constructive mathematics, with contributions from leading specialists surveying the subject’s myriad aspects. Major themes include: constructive algebra and geometry, constructive analysis, constructive topology, constructive logic and foundations of mathematics, and computational aspects of constructive mathematics. A series of introductory chapters provides graduate students and other newcomers to the subject with foundations for the surveys that follow. Edited by four of the most eminent experts in the field, this is an indispensable reference for constructive mathematicians and a fascinating vista of modern constructivism for the increasing number of researchers interested in constructive approaches.

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Preface

Constructive mathematics, in which ‘there exists’ is interpreted strictly as ‘we can find/construct/compute’, can be traced back at least to Kronecker and was first taken up systematically by Brouwer [6] and his ‘intuitionist’ followers. For various reasons, Brouwer’s intuitionistic mathematics (INT), other than its underlying intuitionistic logic, garnered relatively little interest outside parts of Europe. In the Soviet Union in the late 1940s, A. A. Markov began a research programme on recursive constructive mathematics (RUSS), in which ‘constructive’ was interpreted as ‘applying recursion theory and intuitionistic logic to analysis’. Markov’s programme, too, failed to convince mathematicians, other than logicians, that it had much significance for the working mathematician.

The tipping point for constructive mathematics was the publication, in 1967, of Errett Bishop’s groundbreaking monograph *Foundations of Constructive Analysis* [3], in which, confounding the predictions of Hilbert and the majority of active research mathematicians, he presented a fully algorithmic development of deep analysis, including functional analysis and measure theory. Moreover, he did so in the natural style of an analyst, resorting to neither the non-classical principles of Brouwer nor Markov’s framework of recursion theory. The key to his development was the use of intuitionistic logic and an informal set theory (one formalisation of which is described in Chapter 2), the former capturing the Brouwer–Heyting–Kolmogorov (BHK) interpretation of the logical connectives and quantifiers; this meant that his work read like normal analysis rather than mathematical logic.

In a certain sense, intuitionistic logic, which is discussed in Chapter 1, is weaker than classical logic: with the former one cannot prove, for example, the law of excluded middle, De Morgan’s law, or even the seemingly trivial limited principle of omniscience, which states that for every binary sequence, either all the terms are 0 or else there exists a term equal to 1. However, as the Curry–Howard isomorphism shows, we can ensure constructivity in mathematics by using intuitionistic

logic. Moreover, we can extract programs from intuitionistic-logic-based proofs (see Part IV).

In the 50-plus years since the appearance of his book, there has been considerable progress in the continuing development of Bishop's analysis [4, 8, 10]; but the constructive banner has also been raised by algebraists [11, 13], topologists [5, 14], researchers into formal set- and type-theoretic foundations for Bishop-style mathematics (BISH) ([1, 2, 12], Chapter 2), and computer scientists working on program extraction from proofs in BISH [7, 15]. Following the initiative of Veldman [17] and Ishihara [9], there is now also a substantial body of research in constructive reverse mathematics, in which theorems and principles of classical, intuitionistic, and (constructive) recursive mathematics are classified constructively by those principles that are necessary and sufficient additions to BISH in order to derive them (see Chapters 23 and 24).

There is another aspect of constructive mathematics that is increasingly regarded as a *sine qua non*: predicativity. This means ensuring that we avoid self-referential, or *impredicative*, definitions such as

$$A \equiv \{n \in \mathbf{N} : \forall S \subset \mathbf{N} \phi(S, n)\},$$

in which the criterion for membership of n in the definiendum A involves universal quantification over all subsets of \mathbf{N} , including A itself. In the past three decades there has been increasing research activity in *formal topology* [14], with its emphasis on predicativity and point-free constructive mathematics. Formal-topological methods are being applied far more widely than the word topology would suggest, with a considerable body of research into point-free methods in analysis (see Chapters 15–17). More recently, the late Fields Medallist Vladimir Voevodsky introduced homotopy type theory, an approach to constructive mathematics that has attracted a great deal of attention. However, given the length of our Handbook, we refer our readers to the comprehensive treatise *Homotopy Type Theory: Univalent Foundations of Mathematics* [16] for more information on Voevodsky's approach.

The aim of this Handbook is two-fold:

- to provide an accessible introduction to constructive mathematics – its foundations (Parts I and V), its practice within mathematics itself (Parts II–IV), and its significance for computation (Part VI);
- to demonstrate how far mathematics can be developed with the requirements of constructivity and predicativity.

We hope that our compilation will encourage mathematicians of all persuasions to appreciate the power, subtlety, and growing reach of the constructive mathematical enterprise.

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