HANDBOOK OF CONSTRUCTIVE MATHEMATICS

Constructive mathematics – mathematics in which 'there exists' always means 'we can construct' – is enjoying a renaissance. Fifty years on from Bishop's groundbreaking account of constructive analysis, constructive mathematics has spread out to touch almost all areas of mathematics and to have profound influence in theoretical computer science. This handbook gives the most complete overview of modern constructive mathematics, with contributions from leading specialists surveying the subject's myriad aspects. Major themes include: constructive algebra and geometry, constructive analysis, constructive topology, constructive logic and foundations of mathematics, and computational aspects of constructive mathematics. A series of introductory chapters provides graduate students and other newcomers to the subject with foundations for the surveys that follow. Edited by four of the most eminent experts in the field, this is an indispensable reference for constructive mathematicians and a fascinating vista of modern constructivism for the increasing number of researchers interested in constructive approaches.

Encyclopedia of Mathematics and Its Applications

This series is devoted to significant topics or themes that have wide application in mathematics or mathematical science and for which a detailed development of the abstract theory is less important than a thorough and concrete exploration of the implications and applications.

Books in the **Encyclopedia of Mathematics and Its Applications** cover their subjects comprehensively. Less important results may be summarized as exercises at the ends of chapters. For technicalities, readers can be referred to the bibliography, which is expected to be comprehensive. As a result, volumes are encyclopedic references or manageable guides to major subjects.

Encyclopedia of Mathematics and its Applications

All the titles listed below can be obtained from good booksellers or from Cambridge University Press. For a complete series listing visit

www.cambridge.org/mathematics

135 V. Berthé and M. Rigo (eds.) *Combinatorics, Automata and Number Theory*
136 A. Kristály, V. D. Radulescu and C. Varga *Variational Principles in Mathematical Physics, Geometry, and Economics*
137 J. Berstel and C. Reutenauer *Noncommutative Rational Series with Applications*
138 B. Courcelle and J. Engelfriet *Graph Structure and Monadic Second-Order Logic*
139 M. Fiedler *Matrices and Graphs in Geometry*
140 N. Vakil *Real Analysis through Modern Infinitesimals*
141 R. B. Paris *Hadamard Expansions and Hyperasymptotic Evaluation*
142 Y. Crama and P. L. Hammer *Boolean Functions*
143 A. Arapostathis, V. S. Borkar and M. K. Ghosh *Ergodic Control of Diffusion Processes*
144 N. Caspard, B. Leclerc and B. Monjardet *Finite Ordered Sets*
145 D. Z. Arov and H. Dym *Bitangential Direct and Inverse Problems for Systems of Integral and Differential Equations*
146 G. Dassios *Ellipsoidal Harmonics*
147 L. W. Beineke and R. J. Wilson (eds.) with O. R. Oellermann *Topics in Structural Graph Theory*
148 L. Berlyand, A. G. Kolpakov and A. Novikov *Introduction to the Network Approximation Method for Materials Modeling*
149 M. Baake and U. Grimm *Aperiodic Order I: A Mathematical Invitation*
150 J. Borwein et al. *Lattice Sums Then and Now*
151 R. Schneider *Convex Bodies: The Brunn–Minkowski Theory (Second Edition)*
152 G. Da Prato and J. Zabczyk *Stochastic Equations in Infinite Dimensions (Second Edition)*
153 D. Hofmann, G. J. Seal and W. Tholen (eds.) *Monoidal Topology*
154 M. Cabrera García and Á. Rodríguez Palacios *Non-Associative Normed Algebras I: The Vidav–Palmer and Gelfand–Naimark Theorems*
155 C. F. Dunkl and Y. Xu *Orthogonal Polynomials of Several Variables (Second Edition)*
156 L. W. Beineke and R. J. Wilson (eds.) with B. Toft *Topics in Chromatic Graph Theory*
157 T. Mora *Solving Polynomial Equation Systems III: Algebraic Solving*
158 T. Mora *Solving Polynomial Equation Systems IV: Buchberger Theory and Beyond*
159 V. Berthé and M. Rigo (eds.) *Combinatorics, Words and Symbolic Dynamics*
160 B. Rubin *Introduction to Radon Transforms: With Elements of Fractional Calculus and Harmonic Analysis*
161 M. Ghergu and S. D. Taliaferro *Isolated Singularities in Partial Differential Inequalities*
162 G. Molica Bisci, V. D. Radulescu and R. Servadei *Variational Methods for Nonlocal Fractional Problems*
163 S. Wagon *The Banach–Tarski Paradox (Second Edition)*
164 K. Broughan *Equivalents of the Riemann Hypothesis I: Arithmetic Equivalents*
165 K. Broughan *Equivalents of the Riemann Hypothesis II: Analytic Equivalents*
166 M. Baake and U. Grimm (eds.) *Aperiodic Order II: Crystallography and Almost Periodicity*
167 M. Cabrera García and Á. Rodríguez Palacios *Non-Associative Normed Algebras II: Representation Theory and the Zel'manov Approach*
168 A. Yu. Khrennikov, S. V. Kozyrev and W. A. Zúñiga-Galindo *Ultrametric Pseudodifferential Equations and Applications*
169 S. R. Finch *Mathematical Constants II*
170 J. Krajícek *Proof Complexity*
171 D. Bulacu, S. Caenepeel, F. Panaite and F. Van *Oystaeyen Quasi-Hopf Algebras*
172 P. McMullen *Geometric Regular Polytopes*
173 M. Aguiar and S. Mahajan *Bimonoids for Hyperplane Arrangements*
174 M. Barski and J. Zabczyk *Mathematics of the Bond Market: A Lévy Processes Approach*
175 T. R. Bielecki, J. Jakubowski and M. Niewęglowski *Structured Dependence between Stochastic Processes*
176 A. A. Borovkov, V. V. Ulyanov and Mikhail Zhitlukhin *Asymptotic Analysis of Random Walks: Light-Tailed Distributions*
177 Y.-K. Chan *Foundations of Constructive Probability Theory*
178 L. W. Beineke, M. C. Golumbic and R. J. Wilson (eds.) *Topics in Algorithmic Graph Theory*
179 H.-L. Gau and P. Y. Wu *Numerical Ranges of Hilbert Space Operators*
180 P. A. Martin *Time-Domain Scattering*
181 M. D. de la Iglesia *Orthogonal Polynomials in the Spectral Analysis of Markov Processes*
182 A. E. Brouwer and H. Van Maldeghem *Strongly Regular Graphs*
183 D. Z. Arov and O. J. Staffans *Linear State/Signal Systems*
184 A. A. Borovkov *Compound Renewal Processes*
185 D. Bridges, H. Ishihara, M. Rathjen and H. Schwichtenberg (eds.) *Handbook of Constructive Mathematics*

ENCYCLOPEDIA OF MATHEMATICS AND ITS APPLICATIONS

Handbook of Constructive Mathematics

Edited by

DOUGLAS BRIDGES
University of Canterbury

HAJIME ISHIHARA
Japan Advanced Institute of Science and Technology

MICHAEL RATHJEN
University of Leeds

HELMUT SCHWICHTENBERG
Ludwig Maximilian University of Munich

CAMBRIDGE
UNIVERSITY PRESS

CAMBRIDGE
UNIVERSITY PRESS

Shaftesbury Road, Cambridge CB2 8EA, United Kingdom

One Liberty Plaza, 20th Floor, New York, NY 10006, USA

477 Williamstown Road, Port Melbourne, VIC 3207, Australia

314–321, 3rd Floor, Plot 3, Splendor Forum, Jasola District Centre, New Delhi - 110025, India

103 Penang Road, #05-06/07, Visioncrest Commercial, Singapore 238467

Cambridge University Press is part of Cambridge University Press & Assessment,
a department of the University of Cambridge.

We share the University's mission to contribute to society through the pursuit of
education, learning and research at the highest international levels of excellence.

www.cambridge.org
Information on this title: www.cambridge.org/9781316510865
DOI: 10.1017/9781009039888

First published 2023

A catalogue record for this publication is available from the British Library.

ISBN 978-1-316-51086-5 Hardback

Contents

Cambridge University Press & Assessment
978-1-316-51086-5 — Handbook of Constructive Mathematics
Edited by Douglas Bridges , Hajime Ishihara , Michael Rathjen , Helmut Schwichtenberg
Frontmatter
More Information

Contributors

Josef Berger *Department of Mathematics, University of Munich*
Ulrich Berger *Department of Computer Science, Swansea University*
Mark Bickford *Department of Computer Science, Cornell University*
Douglas Bridges *School of Mathematics & Statistics, University of Canterbury*
Yuen-Kwok Chan *Mortgage Analytics, Citigroup (retired)*
Francesco Ciraulo *Department of Mathematics, University of Padua*
Thierry Coquand *Computer Science Department, University of Gothenburg*
Laura Crosilla *Department of Philosophy, IFIKK, University of Oslo*
Robin S. Havea *Tonga Campus, University of the South Pacific*
Matthew Hendtlass *School of Mathematics & Statistics,University of Canterbury*
Hajime Ishihara *School of Information Science, Japan Advanced Institute of Science and Technology*
Gerhard Jäger *Institute of Computer Science, University of Bern*
Tatsuji Kawai *Japan Advanced Institute of Science and Technology, Asahidai*
Davorin Lešnik *Faculty of Mathematics and Physics, University of Ljubljana*
Henri Lombardi *Department of Mathematics, University of Franche-Comté*
Robert S. Lubarsky *Department of Mathematical Sciences, Florida Atlantic University*
Maria Emilia Maietti *Department of Mathematics, University of Padua*
Mark Mandelkern *Department of Mathematics, New Mexico State University*
Kenji Miyamoto *Mathematics Institute, Ludwig Maximilian University of Munich*
Takako Nemoto *Department of Architectural Design, Hiroshima Institute of Technology*
Iosif Petrakis *Mathematics Institute, Ludwig Maximilian University of Munich*
Michael Rathjen *Department of Pure Mathematics, University of Leeds*
Fred Richman *Department of Mathematics, Florida Atlantic University*
Giovanni Sambin *Department of Mathematics, University of Padua*

Cambridge University Press & Assessment
978-1-316-51086-5 — Handbook of Constructive Mathematics
Edited by Douglas Bridges , Hajime Ishihara , Michael Rathjen , Helmut Schwichtenberg
Frontmatter
More Information

Helmut Schwichtenberg *Mathematics Institute, Ludwig Maximilian University of Munich*
Monika Seisenberger *Department of Computer Science, Swansea University*
Gregor Svindland *Institute of Actuarial and Financial Mathematics, Leibniz University Hanover*
Ihsen Yengui *Department of Mathematics, University of Sfax*

Preface

Constructive mathematics, in which 'there exists' is interpreted strictly as 'we can find/construct/compute', can be traced back at least to Kronecker and was first taken up systematically by Brouwer [6] and his 'intuitionist' followers. For various reasons, Brouwer's intuitionistic mathematics (INT), other than its underlying intuitionistic logic, garnered relatively little interest outside parts of Europe. In the Soviet Union in the late 1940s, A. A. Markov began a research programme on recursive constructive mathematics (RUSS), in which 'constructive' was interpreted as 'applying recursion theory and intuitionistic logic to analysis'. Markov's programme, too, failed to convince mathematicians, other than logicians, that it had much significance for the working mathematician.

The tipping point for constructive mathematics was the publication, in 1967, of Errett Bishop's groundbreaking monograph *Foundations of Constructive Analysis* [3], in which, confounding the predictions of Hilbert and the majority of active research mathematicians, he presented a fully algorithmic development of deep analysis, including functional analysis and measure theory. Moreover, he did so in the natural style of an analyst, resorting to neither the non-classical principles of Brouwer nor Markov's framework of recursion theory. The key to his development was the use of intuitionistic logic and an informal set theory (one formalisation of which is described in Chapter 2), the former capturing the Brouwer–Heyting–Kolmogorov (BHK) interpretation of the logical connectives and quantifiers; this meant that his work read like normal analysis rather than mathematical logic.

In a certain sense, intuitionistic logic, which is discussed in Chapter 1, is weaker than classical logic: with the former one cannot prove, for example, the law of excluded middle, De Morgan's law, or even the seemingly trivial limited principle of omniscience, which states that for every binary sequence, either all the terms are 0 or else there exists a term equal to 1. However, as the Curry–Howard isomorphism shows, we can ensure constructivity in mathematics by using intuitionistic logic. Moreover, we can extract programs from intuitionistic-logic-based proofs (see Part IV).

In the 50-plus years since the appearance of his book, there has been considerable progress in the continuing development of Bishop's analysis [4, 8, 10]; but the constructive banner has also been raised by algebraists [11, 13], topologists [5, 14], researchers into formal set- and type-theoretic foundations for Bishop-style mathematics (BISH) ([1, 2, 12], Chapter 2), and computer scientists working on program extraction from proofs in BISH [7, 15]. Following the initiative of Veldman [17] and Ishihara [9], there is now also a substantial body of research in constructive reverse mathematics, in which theorems and principles of classical, intuitionistic, and (constructive) recursive mathematics are classified constructively by those principles that are necessary and sufficient additions to BISH in order to derive them (see Chapters 23 and 24).

There is another aspect of constructive mathematics that is increasingly regarded as a *sine qua non*: predicativity. This means ensuring that we avoid self-referential, or *impredicative*, definitions such as

$$A \equiv \{n \in \mathbf{N} : \forall S \subset \mathbf{N}\, \phi(S, n)\},$$

in which the criterion for membership of n in the definiendum A involves universal quantification over all subsets of $\mathbf{N}$, including A itself. In the past three decades there has been increasing research activity in *formal topology* [14], with its emphasis on predicativity and point-free constructive mathematics. Formal-topological methods are being applied far more widely than the word topology would suggest, with a considerable body of research into point-free methods in analysis (see Chapters 15–17). More recently, the late Fields Medallist Vladimir Voyevodsky introduced homotopy type theory, an approach to constructive mathematics that has attracted a great deal of attention. However, given the length of our Handbook, we refer our readers to the comprehensive treatise *Homotopy Type Theory: Univalent Foundations of Mathematics* [16] for more information on Voyevodsky's approach.

The aim of this Handbook is two-fold:

- to provide an accessible introduction to constructive mathematics – its foundations (Parts I and V), its practice within mathematics itself (Parts II–IV), and its significance for computation (Part VI);
- to demonstrate how far mathematics can be developed with the requirements of constructivity and predicativity.

We hope that our compilation will encourage mathematicians of all persuasions to appreciate the power, subtlety, and growing reach of the constructive mathematical enterprise.

Acknowledgements

The editors wish to thank the following.

Iosif Petrakis, for initiating the handbook project and assisting with its development.

The anonymous referees of our chapters.

Tom Harris and David Tranah, from Cambridge University Press, for their patient guidance over the preparation and production of the Handbook.

The Hausdorff Research Institute of Mathematics, Bonn, for the Trimester *Types, Sets and Constructions* (May–August, 2018).

The European Union, for the projects *Computing with Infinite Data* (2017–2021) and *Correctnesss by Construction* (2014–2017).

The Japan Society for the Promotion of Science, Core-to-Core Program (A. Advanced Research Networks), for the project *Mathematical Logic and its Applications* (2015–2020).

The John Templeton Foundation, for the project *A New Dawn of Intuitionism: Mathematical and Philosophical Advances* (2017–2020).

References

[1] Aczel, P., and Rathjen, M. 2001. *Notes on Constructive Mathematics*. Technical report 40. Royal Swedish Academy of Sciences.

[2] Alps, R. A., and Bridges, D. S. *Morse Set Theory as a Foundation for Constructive Mathematics*. Monograph in preparation.

[3] Bishop, E. 1967. *Foundations of Constructive Analysis*. New York: McGraw-Hill.

[4] Bridges, D. S., and Vîţă, L. S. 2006. *Techniques of Constructive Analysis*. Springer.

[5] Bridges, D. S., and Vîţă, L. S. 2011. *Apartness and Uniformity: A Constructive Development*. CiE series Theory and Applications of Computability. Springer-Verlag.

[6] Brouwer, L. E. J. 1907. Over de Grondslagen der Wiskunde. PhD Thesis, University of Amsterdam.

[7] Constable, R. L., et al. 1986. *Implementing Mathematics with the NUPRL Proof Development System*. Englewood Cliffs, New Jersey: Prentice-Hall.

[8] Ishihara, H. 2001. Locating subsets of a Hilbert space. *Proc. Amer. Math. Soc.*, **129**(5), 1385–2390.

[9] Ishihara, H. 2005. Constructive reverse mathematics: compactness properties. Pages 245–267 of: *From Sets and Types to Topology and Analysis*. Oxford Logic Guides, vol. 48. Oxford: Clarendon Press.

[10] Ishihara, H., and Vîţă, L. S. 2003. Locating subsets of a normed space. *Proc. Amer. Math. Soc.*, **131**(10), 3231–3239.
[11] Lombardi, H., and Quitté, C. 2011. *Algèbre Commutative, Méthodes constructives (Modules projectifs de type fini)*. Montrouge, France: Calvage et Mounet. English translation, with additions and corrections: *Commutative Algebra, Constructive Methods (Finitely Generated Projective Modules)*, Springer-Verlag, 2015.
[12] Martin-Löf, P. 1998. An intuitionistic theory of types. Pages 127–172 of: *Twenty-five Years of Constructive Type Theory*. Oxford Logic Guides, vol. 36. Oxford: Clarendon Press.
[13] Mines, R., Richman F., and Ruitenburg, W. 1988. *A Course in Constructive Algebra*. Universitext. Heidelberg: Springer-Verlag.
[14] Sambin, G. 2023. *Positive Topology and the Basic Picture. New Mathematics Emerging from Dynamic Constructivism*. Oxford Logic Guides. Oxford: Oxford University Press. (In the press.)
[15] Schwichtenberg, H. 2009. Program extraction in constructive analysis. Pages 255–275 of: *Logicism, Intuitionism, and Formalism – What has Become of them?* Synthese Library, vol. 341. Berlin: Springer-Verlag.
[16] Univalent Foundations Program. 2013. *Homotopy Type Theory: Univalent Foundations of Mathematics*. Princeton, New Jersey: Institute for Advanced Study. Available at `https://homotopytypetheory.org/book`.
[17] Veldman, W. 2001. Brouwer's Fan Theorem as an axiom and as a contrast to Kleene's Alternative. *Arch. Math. Logic*, **5–6**, 621–693.