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Excerpt
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Notation, Concepts, and Conventions in Relativity Theory

This chapter serves to briefly review the concepts relevant to the problems presented in this book. Its purpose is to remind the reader of the basic concepts as well as to introduce the notations and conventions that will be used. In particular, some notations and conventions will vary throughout the different textbooks available on the subject. Some of the different notations have been deliberately used in a number of problems in order to familiarize the reader with the fact that different notations occur in the literature.

General Notation

The components of a *vector* V will be written as V^μ in *contravariant* form and V_μ in *covariant* form with the index μ running over all the spacetime coordinates. When referring to 3-vectors, Latin letters will be used for the spatial indices rather than Greek ones, which we use for spacetime coordinates. In the case when an explicit basis for a given vector space is needed, we will use the *partial derivatives* ∂_μ to denote such a basis, i.e.,

$$V = V^\mu \partial_\mu. \tag{0.1}$$

Similarly, *tensor* components will be denoted with superscripts for contravariant indices and subscripts for covariant indices. Tensors with n indices, all down, are called *covariant tensors of rank n* and tensors with n indices, all up, are called *contravariant tensors of rank n*. Tensors with indices both up and down are so-called *mixed tensors*. Thus, a vector is a tensor of rank 1 (one index) and a scalar is a tensor of rank zero (no indices). The *Einstein summation convention* is used throughout the book, implying that indices that are repeated are to be summed over the relevant range. For example, in a four-dimensional spacetime, we have

$$V^\mu U_\mu \equiv \sum_{\mu=0}^{3} V^\mu U_\mu = V^0 U_0 + V^1 U_1 + V^2 U_2 + V^3 U_3, \tag{0.2}$$

where U and V are vectors. Contravariant and covariant components are related by lowering and raising with the metric tensor $g = (g_{\mu\nu})$ and its inverse $g^{-1} = (g^{\mu\nu})$, respectively, i.e.,

$$V_\mu = g_{\mu\nu} V^\nu \qquad \text{and} \qquad V^\mu = g^{\mu\nu} V_\nu. \tag{0.3}$$

Partial derivatives of a given function f may be denoted in several ways, e.g.,

$$\frac{\partial f}{\partial x^\mu} = \partial_\mu f = f_{,\mu}. \tag{0.4}$$

Several indices after the comma in the latter notation represent higher-order derivatives and the notation may also be used for vector components, for which indices belonging to the vector component are written before the comma and indices denoting derivatives after the comma, i.e.,

$$\partial_\mu \partial_\nu f = f_{,\mu\nu} \qquad \text{and} \qquad \partial_\mu V_\nu = V_{\nu,\mu}. \tag{0.5}$$

Objects with two indices may be represented in matrix form. We will indicate this by putting parentheses around the considered objects. For example, we can write the object A with two indices as

$$A = (A_{\mu\nu}) = \begin{pmatrix} A_{00} & A_{01} & A_{02} & A_{03} \\ A_{10} & A_{11} & A_{12} & A_{13} \\ A_{20} & A_{21} & A_{22} & A_{23} \\ A_{30} & A_{31} & A_{32} & A_{33} \end{pmatrix}. \tag{0.6}$$

By convention, the first index represents the row of the matrix and the second index represents the column. When this is used for one covariant index and one contravariant index, the contravariant index is taken as the row index and the covariant index as the column index.

For objects with more than two indices, we may use matrix notation to represent parts of such objects by inserting a bullet ('•') in place of the indices being considered. For example, the components $A^\mu_{1\nu}$ of the three-index object $A^\mu_{\lambda\nu}$ would be represented as the matrix

$$A^{\bullet}_{1\bullet} = (A^\mu_{1\nu}) = \begin{pmatrix} A^0_{10} & A^0_{11} & A^0_{12} & A^0_{13} \\ A^1_{10} & A^1_{11} & A^1_{12} & A^1_{13} \\ A^2_{10} & A^2_{11} & A^2_{12} & A^2_{13} \\ A^3_{10} & A^3_{11} & A^3_{12} & A^3_{13} \end{pmatrix}. \tag{0.7}$$

Special Relativity

In a flat $1 + 3$-dimensional spacetime and in Cartesian coordinates, the *Minkowski metric* is given by

$$ds^2 = \eta_{\mu\nu} dx^\mu dx^\nu = c^2 dt^2 - dx^2 - dy^2 - dz^2, \tag{0.8}$$

where c is the speed of light in vacuum and $x^0 = ct$. In units of $c = 1$, so-called *natural units*, it holds that $x^0 = t$. The *metric tensor* and its inverse, i.e., the *inverse metric tensor*, can be written as

$$\eta = (\eta_{\mu\nu}) = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & -1 \end{pmatrix} \quad \Leftrightarrow \quad \eta^{-1} = (\eta^{\mu\nu}) = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & -1 \end{pmatrix}. \tag{0.9}$$

For any two vectors $x = (x^\mu) = (x^0, x^1, x^2, x^3)$ and $y = (y^\nu) = (y^0, y^1, y^2, y^3)$ in *Minkowski space* described by their contravariant components expressed in Cartesian coordinates, the *Minkowski inner product* is introduced as

$$x \cdot y \equiv x^0 y^0 - x^1 y^1 - x^2 y^2 - x^3 y^3, \tag{0.10}$$

which is obviously commutative, *i.e.*, $x \cdot y = y \cdot x$. We also define the notation $x^2 \equiv x \cdot x = (x^0)^2 - (x^1)^2 - (x^2)^2 - (x^3)^2$ for the squared norm ('length') of the vector x, which is indefinite, since it can be either positive or negative.[1] The Minkowski metric η and its inverse η^{-1} fulfill the relation

$$\eta_{\mu\lambda} \eta^{\lambda\nu} = \eta_\mu{}^\nu = \eta^\nu{}_\mu = \delta^\nu_\mu, \tag{0.11}$$

where δ^ν_μ is the *Kronecker delta* such that $\delta^\nu_\mu = 1$ if $\mu = \nu$ and $\delta^\nu_\mu = 0$ if $\mu \neq \nu$. We can write the Minkowski inner product in multiple ways as

$$x \cdot y = x^\mu \eta_{\mu\nu} y^\nu = \eta_{\mu\nu} x^\mu y^\nu = x^\mu y_\mu = x^0 y_0 + x^i y_i = x^0 y^0 - x^i y^i, \tag{0.12}$$

where, e.g., x^μ can be considered as the contravariant components of the vector x and y_μ the covariant components of the vector y, i.e., $y_0 = y^0$ and $y_i = -y^i$, and it also holds that $x^\mu y_\mu = x_\nu y^\nu$. Furthermore, we say that the vector x is *timelike* if $x^2 > 0$, *lightlike* if $x^2 = 0$, and *spacelike* if $x^2 < 0$. Note that lightlike vectors x form a cone $(x^0)^2 = (x^1)^2 + (x^2)^2 + (x^3)^2$ and a nonspacelike vector x is future pointing if $x^0 > 0$ and past pointing if $x^0 < 0$.

[1] Note the abuse of notation – the symbol x^2 denotes both the 'length' of the vector x and the second spatial contravariant component of the vector x. Unfortunately, this type of abuse of notation is difficult to avoid in relativity theory, since the notation would otherwise be too cumbersome.

In general, a *Lorentz transformation* Λ between two coordinate systems S and S' described by coordinates x and x', respectively, is given by

$$x' = \Lambda x \quad \Leftrightarrow \quad x'^{\mu} = \Lambda^{\mu}{}_{\nu} x^{\nu}. \tag{0.13}$$

In particular, if the Lorentz transformation is a boost in the x^1-direction, we can write

$$\Lambda^{(01)} = \begin{pmatrix} \cosh\theta & -\sinh\theta & 0 & 0 \\ -\sinh\theta & \cosh\theta & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} = \begin{pmatrix} \gamma & -\beta\gamma & 0 & 0 \\ -\beta\gamma & \gamma & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}, \tag{0.14}$$

where θ is the *rapidity*, $\beta \equiv v/c$, and γ is the so-called *gamma factor*, i.e., $\gamma \equiv \gamma(v) \equiv (1-v^2/c^2)^{-1/2}$, with v being the relative speed between the coordinate systems S and S'. Furthermore, it holds that

$$\cosh\theta = \gamma = \frac{1}{\sqrt{1-v^2/c^2}}, \quad \sinh\theta = \beta\gamma = \frac{v}{c}\frac{1}{\sqrt{1-v^2/c^2}}, \quad \tanh\theta = \beta = \frac{v}{c}. \tag{0.15}$$

The formulas for *(Lorentz) length contraction* and *time dilation* are given by

$$\ell' = \frac{\ell}{\gamma(v)} = \ell\sqrt{1-v^2/c^2}, \quad t = t'\gamma(v) = \frac{t'}{\sqrt{1-v^2/c^2}}, \tag{0.16}$$

respectively.

The *relativistic energy–momentum dispersion* relation is given by

$$E' = m\gamma(v)c^2 = \frac{mc^2}{\sqrt{1-v^2/c^2}}, \tag{0.17}$$

where m is the mass of an object, v its speed, and E its energy. In the rest frame of the object, it leads to *Einstein's famous formula*

$$E = mc^2. \tag{0.18}$$

The *relativistic addition of velocities* for colinear velocities v and v' is given by

$$v'' = \frac{v+v'}{1+vv'/c^2}. \tag{0.19}$$

In the nonrelativistic limit, i.e., $v, v' \ll c$, the classical formula $v'' \simeq v + v'$ is recovered.

Consider radiation of light in a specific coordinate direction of the coordinate system S. One should think of the radiation as coming from a fixed source in this coordinate system, where the radiation has frequency ν. For an observer in a coordinate system S' moving along the same coordinate direction with the relative velocity v, a frequency ν' is observed that is given by the *relativistic Doppler formula*, i.e.,

$$\nu' = \nu\sqrt{\frac{c-v}{c+v}}. \tag{0.20}$$

If the observer is moving away from the source, there is a *redshift* in the frequency of light, whereas if the observer is moving toward the source, there is a corresponding *blueshift*.

For the (binary) reaction $A + B \longrightarrow a + b + \cdots$, where two particles with 4-momenta p_A and p_B collide, using *conservation of 4-momentum*, we have

$$p_A + p_B = p_a + p_b + \cdots, \tag{0.21}$$

whereas for the decay $A \longrightarrow a + b + \cdots$, we have the simpler relation

$$p_A = p_a + p_b + p_c + \cdots. \tag{0.22}$$

This can be generalized to any number of particles with corresponding 4-momenta before and after a reaction, i.e.,

$$P_{in} = \sum_{i=A,B,\ldots} p_i = \sum_{j=a,b,\ldots} p_j = P_{out}. \tag{0.23}$$

Note that P^2 is invariant for any $P = \sum_{k=1}^{N} p_k$, where N is the number of particles, and actually, for any two 4-vectors A and B, the Minkowski inner product $A \cdot B = \eta_{\mu\nu} A^\mu B^\nu = A^\mu B_\mu$ is invariant under Lorentz transformations. Especially, $A^2 = A^\mu A_\mu$ is invariant. This is useful in many applications.

In electromagnetism, the *electromagnetic field strength tensor* F is defined as

$$F^{\mu\nu} = \partial^\mu A^\nu - \partial^\nu A^\mu, \tag{0.24}$$

where $A = (A^\mu) = (\phi, c\boldsymbol{A})$ is the 4-vector potential with ϕ and $\boldsymbol{A} = \boldsymbol{A}(x)$ being the electric scalar potential and the magnetic 3-vector potential, respectively, and can be written as

$$F = (F^{\mu\nu}) = \begin{pmatrix} 0 & -E^1 & -E^2 & -E^3 \\ E^1 & 0 & -cB^3 & cB^2 \\ E^2 & cB^3 & 0 & -cB^1 \\ E^3 & -cB^2 & cB^1 & 0 \end{pmatrix}, \tag{0.25}$$

which is a real antisymmetric matrix, i.e., $F^{\mu\nu} = -F^{\nu\mu}$, that combines both the electric and magnetic field strengths, i.e., $\boldsymbol{E} = (E^1, E^2, E^3)$ and $\boldsymbol{B} = (B^1, B^2, B^3)$. Using this tensor, *Maxwell's equations* can be written as

$$\partial_\mu F^{\mu\nu} = j^\nu, \quad \partial^\mu F^{\nu\lambda} + \partial^\nu F^{\lambda\mu} + \partial^\lambda F^{\mu\nu} = 0, \tag{0.26}$$

where $j = (j^\mu) = (\rho, \boldsymbol{j})$ is the 4-current density with $\rho = \rho(x)$ and $\boldsymbol{j} = \boldsymbol{j}(x)$ being the charge density and the electric 3-current density, respectively. In addition, we have the two Lorentz invariants

$$F^{\mu\nu}F_{\mu\nu} = 2\left(c^2\boldsymbol{B}^2 - \boldsymbol{E}^2\right), \quad \epsilon_{\mu\nu\lambda\omega}F^{\mu\nu}F^{\lambda\omega} = -8c\boldsymbol{B}\cdot\boldsymbol{E}, \tag{0.27}$$

where $\epsilon_{\mu\nu\lambda\omega}$ is the *Levi-Civita tensor* with $\epsilon^{0123} = -\epsilon_{0123} = 1$. Maxwell's equations describe how sources (charges and currents) give rise to electric and magnetic fields. Assuming a moving test charge q with rest mass m and parametrizing the trajectory of the test charge as $x = x(s)$, where s is the proper time parameter, the *Lorentz force law* describes how the field strengths determine the trajectory of the test charge and is given by

$$mc^2\ddot{x}^\mu(s) = q\dot{x}_\nu(s)F^{\mu\nu}(x(s)), \tag{0.28}$$

which is covariant under Lorentz transformations. The *energy–momentum tensor* T of the electromagnetic field is defined as

$$T^{\mu\nu} = \epsilon_0 F^\mu{}_\lambda F^{\lambda\nu} + \frac{\epsilon_0}{4}\eta^{\mu\nu}F_{\lambda\omega}F^{\lambda\omega}, \tag{0.29}$$

where ϵ_0 is the electric constant (or permittivity of free space). It holds that T is symmetric, i.e., $T^{\mu\nu} = T^{\nu\mu}$, and $T^\mu{}_\mu = \eta_{\mu\nu}T^{\mu\nu} = 0$. Furthermore, using Maxwell's equations, we obtain

$$\partial_\mu T^{\mu\nu} = \epsilon_0 j_\mu F^{\mu\nu} = -f^\nu, \tag{0.30}$$

where $f = (f^\mu) = (\boldsymbol{j}\cdot\boldsymbol{E}/c, \rho\boldsymbol{E} + \boldsymbol{j}\times\boldsymbol{B})$ is the Lorentz force density generated by the 4-current j. Without (external) sources, i.e., when $j = 0$, T is conserved, i.e., $\partial_\mu T^{\mu\nu} = 0$.

General Relativity

In a curved spacetime, the *metric* is defined as

$$ds^2 = g_{\mu\nu}dx^\mu dx^\nu, \tag{0.31}$$

where the *metric tensor* and its inverse, i.e., the *inverse metric tensor*, are given by

$$g = (g_{\mu\nu}), \quad g^{-1} = (g^{\mu\nu}), \tag{0.32}$$

respectively. In the special case that the spacetime is flat (see Special Relativity), we obtain

$$g_{\mu\nu} = \eta_{\mu\nu}, \quad g^{\mu\nu} = \eta^{\mu\nu}, \tag{0.33}$$

in Minkowski coordinates.

The *covariant derivatives* of a covariant vector A_ν and a contravariant vector A^ν are given by

$$\nabla_\mu A_\nu = A_{\nu;\mu} = \partial_\mu A_\nu - \Gamma^\lambda_{\mu\nu} A_\lambda, \quad \nabla_\mu A^\nu = A^\nu_{;\mu} = \partial_\mu A^\nu + \Gamma^\nu_{\mu\lambda} A^\lambda, \tag{0.34}$$

respectively. In particular, it holds that $\nabla_\mu \partial_\nu = \Gamma^\lambda_{\mu\nu}\partial_\lambda$, where the coefficients $\Gamma^\lambda_{\mu\nu}$ are called the *Christoffel symbols* of the second kind. Given a metric $g_{\mu\nu} = g_{\nu\mu}$, the Christoffel symbols of the Levi-Civita connection can be directly computed from

$$\Gamma^\lambda_{\mu\nu} = \frac{1}{2} g^{\lambda\omega} \left(\partial_\mu g_{\nu\omega} + \partial_\nu g_{\mu\omega} - \partial_\omega g_{\mu\nu} \right). \tag{0.35}$$

In addition, it holds that $\Gamma^\lambda_{\mu\nu} = \Gamma^\lambda_{\nu\mu}$, i.e., the Christoffel symbols are always symmetric with respect to the two lower indices.

The *parallel transport equation* for a vector A^λ is given by

$$\dot{x}^\mu \nabla_\mu A^\lambda = \dot{A}^\lambda + \Gamma^\lambda_{\mu\nu} \dot{x}^\mu A^\nu = 0, \tag{0.36}$$

where the dot above ('˙') denotes differentiation with respect to the curve parameter s. Furthermore, considering the Lagrangian density given by

$$\mathcal{L} = g_{\mu\nu} \dot{x}^\mu \dot{x}^\nu, \tag{0.37}$$

and using the Euler–Lagrange equations, i.e.,

$$\frac{\partial \mathcal{L}}{\partial x^\mu} - \frac{d}{ds} \frac{\partial \mathcal{L}}{\partial \dot{x}^\mu} = 0, \tag{0.38}$$

where $\mu = 0, 1, \ldots, n$, we obtain the *geodesic equations* as

$$\ddot{x}^\lambda + \Gamma^\lambda_{\mu\nu} \dot{x}^\mu \dot{x}^\nu = 0. \tag{0.39}$$

Given three vector fields X, Y, and Z, the *torsion* T and the *curvature* R are defined as

$$T(X,Y) = \nabla_X Y - \nabla_Y X - [X,Y], \tag{0.40}$$

$$R(X,Y)Z = [\nabla_X, \nabla_Y] Z - \nabla_{[X,Y]} Z. \tag{0.41}$$

Both the torsion and curvature are tensors and are therefore linear in all of the arguments X, Y, and Z, including when the arguments are multiplied by a scalar function f, e.g.,

$$T(fX,Y) = T(X,fY) = fT(X,Y), \quad T(X,Y+Z) = T(X,Y) + T(X,Z). \tag{0.42}$$

Furthermore, the torsion and curvature tensors are antisymmetric in the arguments X and Y as defined above

$$T(X,Y) = -T(Y,X) \qquad \text{and} \qquad R(X,Y)Z = -R(Y,X)Z. \tag{0.43}$$

In local coordinates, we have

$$T(\partial_\mu, \partial_\nu)^\lambda \partial_\lambda = T^\lambda_{\mu\nu} \partial_\lambda, \quad R(\partial_\mu, \partial_\nu)\partial_\lambda = R^\omega{}_{\lambda\mu\nu}\partial_\omega, \tag{0.44}$$

and the components of the *torsion tensor* and the *Riemann curvature tensor* may be computed as

$$T^\lambda_{\mu\nu} = \Gamma^\lambda_{\mu\nu} - \Gamma^\lambda_{\nu\mu}, \tag{0.45}$$
$$R^\omega{}_{\lambda\mu\nu} = \partial_\mu \Gamma^\omega_{\nu\lambda} - \partial_\nu \Gamma^\omega_{\mu\lambda} + \Gamma^\omega_{\mu\rho}\Gamma^\rho_{\nu\lambda} - \Gamma^\omega_{\nu\rho}\Gamma^\rho_{\mu\lambda}, \tag{0.46}$$

respectively. Note that the Levi-Civita connection is torsion free as $\Gamma^\lambda_{\mu\nu} = \Gamma^\lambda_{\nu\mu}$. For fixed μ and ν, we can write the Riemann curvature tensor in matrix form as

$$R^\bullet{}_{\bullet\mu\nu} = \partial_\mu \Gamma^\bullet_{\nu\bullet} - \partial_\nu \Gamma^\bullet_{\mu\bullet} + \left[\Gamma^\bullet_{\mu\bullet}, \Gamma^\bullet_{\nu\bullet}\right]. \tag{0.47}$$

Note that the Riemann curvature tensor is antisymmetric in μ and ν, i.e., $R^\bullet{}_{\bullet\mu\nu} = -R^\bullet{}_{\bullet\nu\mu}$, or in component form, $R^\omega{}_{\lambda\mu\nu} = -R^\omega{}_{\lambda\nu\mu}$. If the torsion vanishes, i.e., $T = 0$, then we have the *first Bianchi identity*, i.e.,

$$R(X,Y)Z + R(Y,Z)X + R(Z,X)Y = 0, \tag{0.48}$$

or in component form, we have

$$R^\omega{}_{\lambda\mu\nu} + R^\omega{}_{\mu\nu\lambda} + R^\omega{}_{\nu\lambda\mu} = 0. \tag{0.49}$$

Furthermore, we have the *second Bianchi identity* in matrix form, i.e.,

$$\partial_\mu R^\bullet{}_{\bullet\nu\lambda} + \left[\Gamma^\bullet_{\mu\bullet}, R^\bullet{}_{\bullet\nu\lambda}\right] + \partial_\nu R^\bullet{}_{\bullet\lambda\mu} + \left[\Gamma^\bullet_{\nu\bullet}, R^\bullet{}_{\bullet\lambda\mu}\right] + \partial_\lambda R^\bullet{}_{\bullet\mu\nu} + \left[\Gamma^\bullet_{\lambda\bullet}, R^\bullet{}_{\bullet\mu\nu}\right] = 0. \tag{0.50}$$

Using the Riemann curvature tensor, the *Ricci tensor* can be defined as

$$R_{\mu\nu} = R^\lambda{}_{\mu\lambda\nu}, \tag{0.51}$$

which is symmetric, i.e., $R_{\mu\nu} = R_{\nu\mu}$, and in turn, the *Ricci scalar* is defined as

$$R = g^{\mu\nu} R_{\mu\nu} = R^\mu_\mu. \tag{0.52}$$

Finally, the *Einstein tensor* is defined in terms of the Ricci tensor, the Ricci scalar, and the metric tensor as

$$G_{\mu\nu} = R_{\mu\nu} - \frac{1}{2} R g_{\mu\nu}. \tag{0.53}$$

Note that it holds that the Einstein tensor is symmetric, i.e., $G_{\mu\nu} = G_{\nu\mu}$, and conserved, i.e., its covariant divergence vanishes $\nabla_\mu G^{\mu\nu} = 0$. Under local coordinate transformations $y = y(x)$, we have

$$g'_{\mu\nu}(y) = \frac{\partial x^\alpha}{\partial y^\mu}\frac{\partial x^\beta}{\partial y^\nu} g_{\alpha\beta}(x), \tag{0.54}$$

$$\Gamma'^{\lambda}_{\mu\nu}(y) = \frac{\partial x^\alpha}{\partial y^\mu}\frac{\partial x^\beta}{\partial y^\nu}\frac{\partial y^\lambda}{\partial x^\gamma}\Gamma^\gamma_{\alpha\beta}(x) + \frac{\partial y^\lambda}{\partial x^\gamma}\frac{\partial^2 x^\gamma}{\partial y^\mu \partial y^\nu}, \tag{0.55}$$

$$T'^{\lambda}_{\mu\nu}(y) = \frac{\partial y^\lambda}{\partial x^\gamma}\frac{\partial x^\alpha}{\partial y^\mu}\frac{\partial x^\beta}{\partial y^\nu} T^\gamma_{\alpha\beta}(x), \tag{0.56}$$

$$R'^{\omega}{}_{\lambda\mu\nu}(y) = \frac{\partial y^\omega}{\partial x^\delta}\frac{\partial x^\gamma}{\partial y^\lambda}\frac{\partial x^\alpha}{\partial y^\mu}\frac{\partial x^\beta}{\partial y^\nu} R^\delta{}_{\gamma\alpha\beta}(x). \tag{0.57}$$

Symmetries of a spacetime metric are associated to so-called *Killing vector fields*. Consider a vector field X. By definition, X is a Killing vector field if

$$\nabla_\mu X_\nu + \nabla_\nu X_\mu = 0, \tag{0.58}$$

for all indices μ and ν. Given a Killing vector field X^μ and a geodesic described by coordinate functions $x^\mu(s)$, the quantity

$$Q = \dot{x}^\mu X_\mu = g_{\mu\nu}\dot{x}^\mu X^\nu, \tag{0.59}$$

is constant along the geodesic.

The dynamics of spacetime in vacuum are described in the Lagrange formalism using the *Einstein–Hilbert action*, namely

$$\mathscr{S}_{\rm EH} = -\frac{M_{\rm Pl}^2}{2}\int R\sqrt{|\bar{g}|}\, d^4x, \tag{0.60}$$

where $M_{\rm Pl} \equiv c^2/\sqrt{8\pi G}$ is the Planck mass, R is the Ricci scalar, and $\bar{g} = \det(g)$ is the determinant of the metric tensor. For the case of a spacetime not in vacuum, a matter contribution to the action is necessary

$$\mathscr{S}_{\rm matter} = \int \mathcal{L}\sqrt{|\bar{g}|}\, d^4x, \tag{0.61}$$

where $\mathcal{L}$ is the Lagrangian density of the matter contribution.

The *Einstein gravitational field equations* (or simply Einstein's equations) follow from the Euler–Lagrange equations for the action and are given by

$$G_{\mu\nu} = \frac{8\pi G}{c^4} T_{\mu\nu}, \tag{0.62}$$

where G is Newton's gravitational constant and $T^{\mu\nu}$ is the *energy–momentum tensor* (or the stress–energy tensor) that describes the distribution of energy in spacetime. The energy–momentum tensor is generally given by

$$T_{\mu\nu} = \frac{2}{\sqrt{|\bar{g}|}}\frac{\delta\mathscr{S}_{\rm matter}}{\delta g^{\mu\nu}}. \tag{0.63}$$

For example, an external electromagnetic field gives a contribution to $T^{\mu\nu}$ such that (see Special Relativity)

$$T^{\mu\nu}_{\rm EM} = \epsilon_0 F^{\mu}{}_{\lambda} F^{\lambda\nu} + \frac{\epsilon_0}{4} g^{\mu\nu} F_{\lambda\omega} F^{\lambda\omega}, \tag{0.64}$$

whereas a *perfect fluid* (characterized by a 4-velocity u, a scalar density ρ_0, and a scalar pressure p) gives

$$T^{\mu\nu}_{\rm pf} = (\rho_0 + p) u^\mu u^\nu - p g^{\mu\nu}. \tag{0.65}$$

In vacuum, Einstein's equations reduce to $G_{\mu\nu} = 0$.

In the Newtonian limit and the weak field approximation, i.e., $g_{\mu\nu} \simeq \eta_{\mu\nu} + h_{\mu\nu}$, where $h_{\mu\nu}$ is a small perturbation, the solutions to Einstein's equations are given by

$$h_{00} = h_{11} = h_{22} = h_{33} = \frac{2}{c^2}\Phi, \quad h_{\mu\nu} = 0 \ \ \forall \mu \neq \nu, \tag{0.66}$$

where Φ is the gravitational potential for the matter distribution ρ and given by $\Phi = -GM/r$, that is the solution to the Newtonian equation $\nabla^2\Phi = 4\pi G\rho$. Furthermore, the geodesic equations of motion become

$$\frac{d^2x^i}{dt^2} = \partial^i \Phi = -\partial_i \Phi. \tag{0.67}$$

The spherically symmetric vacuum solution to Einstein's equations is the *Schwarzschild solution* for which the *Schwarzschild metric* in spherical coordinates is given by

$$ds^2 = g_{\mu\nu} dx^\mu dx^\nu = \left(1 - \frac{2GM}{c^2 r}\right) c^2 dt^2 - \left(1 - \frac{2GM}{c^2 r}\right)^{-1} dr^2 - r^2 d\Omega^2, \tag{0.68}$$

where $d\Omega^2$ describes the metric on a sphere, i.e., $d\Omega^2 = d\theta^2 + \sin^2\theta d\phi^2$. For large r, the Schwarzschild metric approaches the Minkowski metric. The particular value $r = r_* \equiv 2GM/c^2$ represents the *Schwarzschild event horizon* (or the Schwarzschild radius) and is a coordinate singularity, i.e., it can be removed by a change of coordinates. Such a coordinate change is given by *Kruskal–Szekeres coordinates* u, v, θ, and ϕ, where θ and ϕ are the ordinary spherical coordinates on a unit sphere $\mathbb{S}^2$, the *Kruskal–Szekeres metric* is given by

$$ds^2 = \frac{16\mu^2}{r} e^{(2\mu - r)/(2\mu)} du dv - r^2 d\Omega^2, \quad uv = (2\mu - r) e^{(r-2\mu)/(2\mu)} < \frac{2GM}{c^2 e}, \tag{0.69}$$

where $\mu \equiv GM/c^2$.