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CHAPTER I

NUMBER

§ 1. FUNDAMENTAL LAWS OF ALGEBRA

1. ALGEBRA may be described as the general science of arithmetic In arithmetic the processes of addition, subtraction, multiplication, division and other derived operations arise and are applied in connection with particular numbers. It is seen that the methods employed, being applicable to all particular numbers encountered, with certain well-defined exceptions, must be capable of a general formulation. It is this general formulation which is the primary object of algebra.

In order to express the truths of arithmetic in the general (algebraic) form it is necessary to employ some kind of *symbolism.* The choice of a symbolism is to some extent arbitrary, but not without effect on the development of the science; the lack of an appropriate symbolism having caused in some instances a delay of hundreds of years in the progress of different branches of mathematics. In the present matter of the algebraic statement of the acts of arithmetic the use of the *letters of the alphabet* to represent the unidentified numbers under consideration is singularly appropriate and has been universal since the sixteenth century. Its fertility is sufficiently apparent in the use of formulae in elementary algebra, and in all branches of science, for it to be unnecessary here to enlarge upon it. It may be noted in passing however that the letters occurring in algebraic theorems do not always represent numbers with the same scope of generality. Thus in some theorems the letters (or some of them) may represent numbers belonging to the widest class contemplated,—that of the real or complex numbers,—including under this head, not only the whole numbers 1, 2, 3, etc., but also such entities as -1, $\frac{1}{2}$, $\sqrt{2}$, $\sqrt{-1}$, which are defined below; or they may be restricted to represent numbers of only a particular class, such as the whole numbers, or proper fractions, etc. On the other hand it is always the method of algebra to state theorems with the greatest possible generality, and the fewer the restrictions placed on the variables (i.e. letters) occurring in a theorem the more important (in general) will the theorem be.

2. Fundamental laws. The theorems on which algebra is built, —*the fundamental laws of algebra*,—are based on our intuitive ideas of *counting*, and therefore in the first instance are stated only for the ordinary *whole numbers* 1, 2, 3, etc. The laws are:—

(I) *The associative law for addition*, viz.: The terms of a sum of three numbers may be added together in any way preserving the original order without altering the sum; or, symbolically, if a, b, c are any three whole numbers then $a+(b+c)=(a+b)+c$.

(II) *The associative law for multiplication*, viz.: The terms of a product of three terms may be multiplied in any way preserving the original order without altering the product, or

$$a\times(b\times c)=(a\times b)\times c,$$

where a, b, c have the same significance as in (I).

(III) *The commutative law for addition*, viz.: The terms in a sum of two numbers may be added together in either order without altering the sum, or $a+b=b+a$.

(IV) *The commutative law for multiplication*, viz.: The terms in a product may be multiplied in either order without altering the product, or $a\times b=b\times a$.

(V) *The distributive law*, viz.: The product of a sum of two numbers by any third number is the sum of the products of the separate terms of the sum by the common multiplier, or

$$(a+b)\times c=(a\times c)+(b\times c).$$

In the statement of these laws we must be quite clear as to the meaning of the terms and symbols used. The simplest definition of the sum $a+b$ is to consider it as the number finally obtained if, having counted up to the number a, we continue the process of counting until an additional stock of b objects is exhausted. Multiplication is repeated addition. Subtraction and division will be the inverses of addition and multiplication respectively.

Probably no argument which could be put forward would make the student more convinced of the truth of (I) and (III) than he already is,—though such argument might prove useful in helping the student to appreciate the difficulties inherent in the foundations of all science. We will therefore be content to assume these laws as axioms. We shall have a few further remarks to make in this connection shortly. We can however convince ourselves of the truth of (II), (IV) and (V) by arguments similar to the following, dealing with the commutative law for multiplication (IV): From the definition of multiplication $a \times b$ is the number of objects in an imagined array consisting of b rows each containing a objects*; by intuition, or by extending the laws (I) and (III) to cover the case of any number of terms, we see that the number of objects in our array is the same in whatever order we count the objects; in particular therefore this number is the same as that obtained by considering the array of objects as a columns of b objects, i.e. $b \times a$; and the theorem is proved.

The fundamental laws can be extended to deal with sums, etc., of any number of terms; these extensions follow from the laws themselves and will provide an excellent exercise for the student. In most algebraic deductions from the fundamental laws it will be found most desirable to employ such simple extensions of the laws instead of the original laws themselves.

3. The fundamental laws are at the base of all algebraical analysis, and in fact bear to algebra much the same relation as do the axioms of Euclid to ordinary elementary geometry. Taking, as suggested above, *counting* as the basis of our definitions, we cannot avoid belief in the truth of the laws;—our intuitive ideas on counting would be at variance with a denial of the fundamental laws of addition; and, having accepted these intuitions, expressed in the form of these laws or in any other equivalent form, all the laws,—and thence briefly all algebra,—follow as a logical consequence. But it is interesting to see whether any contrary laws are logically possible, that is, whether, either by distrusting our intuitions or by adopting other more or less arbitrary definitions of "addition," etc., we could without contradiction build up a system of "algebra" on the supposition that our fundamental laws (or some of them) were untrue. In geometry it is possible to deny the Euclidean system of axioms and build up perfectly logical non-Euclidean geometries,—in which for example it is no longer impossible

* The symbol $a \times b$ is to be read as "a multiplied by b"; and in general in any such arithmetical symbol the operations occurring are to be performed successively from the left to the right, except when brackets or special conventions otherwise direct.

for two straight lines to enclose a space. In algebra likewise the answer to this question is in the affirmative. There are in fact, for example, *non-commutative* algebras, i.e. algebras in which the symbols $a \times b$ and $b \times a$ (say) are not equivalent. In such systems the symbols used cannot represent the same entities as in the ordinary algebra of whole numbers unless we are prepared to deny our intuitions of counting. But by using the symbols to represent different operations and combinations of operations of a more general character than our elementary addition, multiplication, etc. (a process to which we have to resort even in ordinary algebra once we pass on to consider the addition, etc., of non-integral numbers), quite definite non-commutative algebras can be obtained.

A trivial example of such a system would be obtained if a were taken to represent a motion of a point through a distance a in an easterly direction on the spherical earth, b a motion through a distance b in a northerly direction, and the multiplication of a by b the motion a followed by the motion b. The results of the two combined motions $a \times b$ and $b \times a$ would then differ by an amount depending on the latitudes of the points considered.

In this course, however, we shall not be concerned with non-commutative or other algebras in which the fundamental laws do not hold. But we shall use these laws as a guide to help us to introduce into algebra and arithmetic entities, other than whole numbers, which will also satisfy these laws and therefore be capable of being dealt with in the same way as the whole numbers.

4. Subtraction and division. If we introduce the notions inverse to those of addition and multiplication, viz., subtraction and division, the fundamental laws are capable of extension to a certain extent. Thus the distributive law (V) will remain true if the sum of the numbers be replaced by the difference, or the product with the third number by the quotient, or both sum and product be replaced by difference and quotient simultaneously, thus e.g. $(a-b) \div c = (a \div c) - (b \div c)$*. The associative and commutative laws cease to hold when addition or multiplication is replaced by subtraction or division; it is known for example that $a-b$ is not the same as $b-a$ and that $a \div (b \div c)$ is not the same as $(a \div b) \div c$. But it is easy to modify the two associative laws so that they will

* It is tacitly assumed here that the operations involved are possible. Thus $a > b$ (i.e. a is greater than b), so that b can be subtracted from a; and so on.

remain true in these cases by simply introducing appropriate *rules of signs*, e.g. by replacing $a-(b+c)$ by $a-b-c$ or $a\div(b\times c)$ by $a\div b\div c$ or $a-(b-c)$ by $a-b+c$*—*provided* the operations considered do not lead at any stage to any impossible operation, i.e. provided, for example, we are not led to the operation of subtracting a number from one not greater than itself nor to the operation of dividing a number by a number of which the first is not a multiple. It will be seen incidentally that any attempt to modify similarly the two commutative laws would, except in trivial cases, essentially lead to some such impossible operation.

EXAMPLES I.

1. Prove the extensions of the fundamental laws to the operations of subtraction and division mentioned above.

2. Prove by imagining m^2 units arranged in a square and adding up in suitable orders that

$$\text{(i)}\quad 1+3+5+\ldots+(2m-1)=m^2,$$

and

$$\text{(ii)}\quad 1+2+3+\ldots+m=\tfrac{1}{2}(m^2+m).$$

3. Prove by direct reference to the fundamental laws that

$$\text{(i)}\quad (a+b)^2=a^2+2ab+b^2,$$

$$\text{(ii)}\quad (a+b)^3=a^3+3a^2b+3ab^2+b^3,$$

$$\text{(iii)}\quad (a+b)(a-b)=a^2-b^2,$$

a and b being any whole numbers (with the restriction that in (iii) a must exceed b); the definitions of the square and cube of a number being supposed known.

4. Prove that if $a>b$ and $c>d$ then

$$(a-b)\times(c-d)=(ac+bd)-(ad+bc).$$

5. Prove that if $a>b$ and $c>d$ then

$$(a-b)(c-d)=ac-ad-bc+bd,$$

provided also $ac>ad+bc$.

[If ac is not greater than $ad+bc$ the expression $ac-ad-bc+bd$ is meaningless as a whole number.]

§ 2. Rational Numbers

5. The inverse operations, subtraction and division, cannot, as we have seen, be applied to all whole numbers indiscriminately. For example (as we are at present dealing exclusively with whole

* To prove for example that $a+(b-c)=a+b-c$ we argue: From the definition of subtraction, if $b-c=x$, $b=x+c$; whence $(a+x)+c=a+(x+c)=a+b$ and therefore $a+(b-c)=a+x=(a+b)-c$. Or to prove $a-(b+c)=a-b-c$, let $a-(b+c)=x$, so that $a=x+(b+c)=(x+c)+b$ whence $a-b=x+c$ whence again $a-b-c=x$.

numbers) 6 cannot be subtracted from 4, because there is no whole number which when added to 6 will give the result 4; and again 7 cannot be divided by 3 because there is no whole number which when multiplied by 3 will give the result 7. The problem confronts us therefore, whether these impossibilities can be surmounted in any way; i.e. whether we can invent new entities, which we may wish to call "numbers," and new operations with these "numbers," which we may call "addition," etc., so that such hitherto impossible operations will be possible in the sense that, considered as an operation in our new "arithmetic," it gives a definite result which is a "number" of the new type. Whether or not such a new arbitrary "arithmetic" can be of any use for practical applications is not primarily the concern of pure mathematics; but as a matter of fact we shall find that our new arithmetic will have useful applications,—in particular to the important problem of measurement.

6. Fractional-number-pairs. Let us define first the *fractional numbers* or *fractions*. We observe first that to some pairs of numbers (e.g. 8 and 2) correspond, by the process of division, single numbers (4), whereas to other pairs (e.g. 7 and 3) there are no such corresponding numbers. For the first kind of pairs of numbers,—such as (8 and 2), or $\frac{8}{2}$,—which represent, or correspond to, whole numbers, we have the following laws of addition, etc.:

If p, q, r, s are whole numbers such that p and r are respectively divisible by q and s, then:

$$\frac{p}{q}+\frac{r}{s}=\frac{ps+qr}{qs}, \qquad \frac{p}{q}\times\frac{r}{s}=\frac{pr}{qs};$$

for (to prove the first relation)

$$\left(\frac{p}{q}+\frac{r}{s}\right)\times qs=\left(\frac{p}{q}\times qs\right)+\left(\frac{r}{s}\times qs\right) \text{ by the distributive law,}$$

$$=\left(\frac{p}{q}\times q\right)\times s+\left(\frac{r}{s}\times s\right)\times q \text{ by the associative law,}$$

$$=(p\times s)+(r\times q) \text{ by the definition of division,}$$

and therefore $$\frac{p}{q}+\frac{r}{s}=\frac{ps+qr}{qs}$$

by the definition of division,—all the numbers concerned being whole numbers.

Or, using the notation (p, q) instead of $\frac{p}{q}$, we have the following laws for the addition and multiplication of pairs of numbers of the type considered:

$$\left.\begin{aligned}(p, q)+(r, s)&=(ps+qr, qs)\\ (p, q)\times(r, s)&=(pr, qs)\end{aligned}\right\} \quad \ldots\ldots\ldots\ldots\ldots(1).$$

In the statement of these laws there is no need for the restriction that the numbers p, r should be divisible by q, s as supposed, i.e. should be such that the various pairs of numbers correspond to (or represent) whole numbers;—*the pairs of numbers*

$$(p, q),\ (r, s),\ (ps+qr, qs),\ (pr, qs)$$

exist equally whether the whole numbers p, r are divisible by the numbers q, s or not. Let us then relax this restriction and use these laws as the *definitions* of "addition" and "multiplication" of "*fractional-number-pairs,*"—or, as we will say for the sake of brevity, *fractional numbers*, or *fractions**.

With these definitions we see easily that the fundamental laws which we have found for addition, etc., of whole numbers, are still formally true. Thus, if A and B are two fractional-number-pairs, say $A=(p, q)$ and $B=(r, s)$, then

$$\begin{aligned}A+B&=(p, q)+(r, s)\\ &=(ps+qr, qs),\end{aligned}$$

and

$$\begin{aligned}B+A&=(r, s)+(p, q)\\ &=(rq+sp, sq)\\ &=(ps+qr, qs),\end{aligned}$$

by the commutative laws for whole numbers. Hence $A+B=B+A$, or the commutative law for addition holds. Similarly the other fundamental laws can be proved to be still true for our "fractional numbers," i.e. if we call a "fractional-number-pair" a *number* and if we call the operations on these fractional-number-pairs defined in equations (1) *addition* and *multiplication*, then the fundamental laws can be applied to these numbers and operations of addition, etc., verbally unchanged. Consequently all general statements about arithmetical operations on whole numbers deducible from the fundamental laws are equally true of these arbitrary operations

* Or positive rational numbers.

on these arbitrary fractional numbers; for example, in particular, if A and B are any two fractional numbers, then

$$(A+B)^2 = A^2 + 2AB + B^2.$$

7. Order. Our arithmetic is not yet the arithmetic of fractional numbers which we desire. The definitions so far given do not enable us to assign any *order* to the fractional numbers, whereas the notion of order is certainly essential in connection with whole numbers. If we wish to use our fractional numbers as numbers in any complete sense it is therefore necessary to give a definition determining which of two fractional numbers is the "greater" and which the "less."

We say that *the fraction* (p, q) *is greater than, equal to, or less than, the fraction* (r, s) *according as the whole number* ps *is greater than, equal to, or less than, the whole number* qr; or, in symbols,

$$\left.\begin{array}{ll} & (p, q) >, =, \text{ or } < (r, s) \\ \text{according as} & ps >, =, \text{ or } < qr \end{array}\right\} \quad \text{.................(2).}$$

In this definition we assume a knowledge of the notions of greater than, etc., as applied to whole numbers.

We notice that from the relations (2) we have

$$(ps, qs) = (p, q) \text{.......................(2}a\text{).}$$

This relation,—which corresponds with the characteristic property of a quotient $\frac{p}{q}$ that $\frac{ps}{qs} = \frac{p}{q}$,—is interesting as being a statement of *equality* between two fractional-number-pairs which are plainly *different* pairs of numbers. The use of the notion and sign of equality in this somewhat arbitrary sense is logically important and interesting, but we will not dwell on it. It is sufficient for us to realise that with this *use* of the sign of equality, whatever it may *mean*, no contradiction can arise.

8. Number-pairs corresponding to whole numbers. We notice the following particular facts about the new arithmetic concerning the *special class* of fractional-number-pairs whose second number is 1:

$$(p, 1) + (q, 1) = (p+q, 1) \text{ and } (p, 1) \times (q, 1) = (pq, 1).$$

It follows that this particular class of number-pairs can be treated for addition and multiplication as a class by itself, for such operations lead only to number-pairs of this kind, having 1 for the second number. In fact a little reflection will shew that in dealing with such special number-pairs we can operate with them just as with the ordinary whole numbers p and q and the ordinary arithmetical operations of addition and multiplication; that in fact the new arithmetic of these special number-pairs is identical with the old arithmetic;—i.e. that *every result in either arithmetic is capable of interpretation as a result in the other*. If therefore in our arithmetic of fractional-number-pairs we agree to replace any number-pair whose second number is 1 by the ordinary whole number which is the first number, i.e. to put

$$(p, 1) = p \quad \text{..........................(3)},$$

the results we shall get will be quite valid, that is to say any algebraic result we obtain for whole numbers will be true as a result in ordinary arithmetic and any result for our fractional-number-pairs will be a true result in our new arithmetic; if in any case the numbers concerned are of both kinds the apparently whole numbers (e.g. p) are to be replaced by the corresponding fractional-number-pair [$(p, 1)$] and the result interpreted as a result in the new arithmetic. Also, applying the relation ($2a$) to these fractional-number-pairs having the second number 1 and coupling it with our convention (3), we obtain the other fundamental characteristic of a fraction that

$$\frac{p}{q} \times q = p;$$

for $$(p, q) \times (q, 1) = (pq, q) = (p, 1).$$

From this it follows that

$$(p, q) = (p, 1)/(q, 1).$$

This last relation is a relation in the new arithmetic and the operation of division is the new arbitrary kind of division (the inverse of the new multiplication).

But we notice that if p is divisible by q (with the quotient r say), —in the ordinary sense,—then the expression $(p, 1)/(q, 1)$ can be replaced by p/q or r; i.e. if p is divisible by q the fractional-number-pair (p, q) represents the ordinary quotient p/q; or, a fractional-number-pair (p, q) for which p is divisible by q behaves as far as arithmetical facts are concerned just like the quotient $\frac{p}{q}$. In future we shall write (p, q) as $\frac{p}{q}$ or p/q, whether p is divisible by q or not.

9. The arithmetic of our fractional-number-pairs may therefore be said to include the arithmetic of such quotients. Our fractional-number-pairs and their "arithmetic" are seen to be the extension of the number system and arithmetic of which we have been in search. In our new system such an operation as $3 \div 8$ is possible.

The advantage of this extended arithmetic from the point of view of the original arithmetic is considerable. We may, in proving a certain result, even in ordinary arithmetic of whole numbers, find it more convenient to use fractional numbers in the intermediate steps of the proof. The proof will nevertheless be a valid proof of the theorem concerned, provided only the final stages of the argument can be interpreted (by the convention (3)) as a relation between whole numbers.

Thus, to take a fundamental, if simple, example: if a, b, and c are whole numbers such that b is a multiple of c and ac a multiple of b, we can argue

$$a \div (b/c) = (a/b) \times c = (ac)/b$$

by using the simple extension to division of the associative law; and in this the expression $(a/b) \times c$ will be meaningless as a whole number unless a is a multiple of b; but nevertheless the result $a \div (b/c) = (ac)/b$ is a true result holding between whole numbers and this proof is quite valid.

10. Subtractive-number-pairs. In the same way we can extend the meaning of subtraction to cases hitherto impossible by introducing *subtractive-number-pairs** $\{p, q\}$ subject to the definitions:

$$\{p, q\} + \{r, s\} = \{p + r, q + s\};$$

$$\{p, q\} \times \{r, s\} = \{pr + qs, ps + qr\};$$

$$\{p, q\} >, =, \text{ or } < \{r, s\}$$

according as

$$p + s >, =, \text{ or } < q + r;$$

$$\{p + q, q\} = p\dagger;$$

where p, q, r, s represent any whole numbers or fractions as hitherto defined.

It is necessary here also to introduce a new number, *zero* or 0, defined as the subtractive-number-pair $\{p, p\}$.

* These are the positive and negative rational numbers.

† This relation could be replaced by $\{p+1, 1\} = p$ and so brought more into agreement with relation (3) of p. 9.