Introduction

Artin-Scheier theory. Let K be a field of characteristic p. Then for every $a \in K$ the polynomial $X^p - X - a$ is separable, and the additive group $\mathbf{F}_p$ acts transitively on the set of zeroes in a field extension. By Artin-Scheier theory, every Galois extension L/K with $\mathrm{Gal}(L/K) = \mathbf{F}_p$ arises as the splitting field of such a polynomial. In other words, there is an exact sequence

$$0 \longrightarrow \mathbf{F}_p \longrightarrow K \xrightarrow{1-\sigma} K \longrightarrow \mathrm{Hom}(\mathrm{Gal}(K^{\mathrm{sep}}/K), \mathbf{F}_p) \longrightarrow 0,$$

where σ is the p-th power Frobenius map. In the language of étale cohomology, this can be rewritten without the choice of a separable closure as an exact sequence

$$0 \longrightarrow \mathbf{F}_p \longrightarrow K \xrightarrow{1-\sigma} K \longrightarrow \mathrm{H}^1((\mathrm{Spec}\, K)_{\mathrm{et}}, \mathbf{F}_p) \longrightarrow 0. \tag{1}$$

More generally, if S is a scheme over $\mathbf{F}_p$, then we have an exact sequence

$$0 \longrightarrow (\mathbf{F}_p)_S \longrightarrow \mathbf{G}_{a,S} \xrightarrow{1-\sigma} \mathbf{G}_{a,S} \longrightarrow 0$$

of sheaves on S_{et}. Here $(\mathbf{F}_p)_S$ denotes the constant sheaf with stalk $\mathbf{F}_p$. Since the étale and Zariski cohomology of $\mathbf{G}_{a,S} = \mathcal{O}_S$ coincide, we obtain a long exact sequence

$$\cdots \longrightarrow \mathrm{H}^i(S_{\mathrm{et}}, \mathbf{F}_p) \longrightarrow \mathrm{H}^i(S, \mathcal{O}_S) \xrightarrow{1-\sigma} \mathrm{H}^i(S, \mathcal{O}_S) \longrightarrow \cdots \tag{2}$$

relating the mod p étale cohomology of S with the coherent cohomology of S, generalizing (1).

Katz and locally constant coefficients. Let S be a noetherian scheme over $\mathbf{F}_p$. In his paper on p-adic properties of modular forms Nick Katz [**35**] showed that there is an equivalence of categories between

(i) pairs $(\mathcal{F}, \tau)$ consisting of a locally free $\mathcal{O}_S$-module $\mathcal{F}$ and an isomorphism $\tau\colon \sigma^\star\mathcal{F} \to \mathcal{F}$ of $\mathcal{O}_S$-modules;
(ii) $\mathbf{F}_p$-modules V on S_{et} that are locally constant of finite rank.

By adjunction the map τ defines an $\mathcal{O}_S$-linear map $\tau_a\colon \mathcal{F} \to \sigma_\star\mathcal{F}$. Since σ is the identity on the underlying topological space of S, we have a natural identification $\sigma_\star\mathcal{F} = \mathcal{F}$. Under this identification τ_a becomes an additive map $\tau_s\colon \mathcal{F} \to \mathcal{F}$ satisfying $\tau_s(fs) = f^p\tau_s(s)$ for all local sections f of $\mathcal{O}_S$ and s of $\mathcal{F}$. (The subscripts a and s to τ stand for *adjoint* and *semi-linear*, respectively.) Any of the three maps τ, τ_a, τ_s determines the other two.

The étale $\mathbf{F}_p$-module V corresponding to $(\mathcal{F}, \tau)$ is defined by a short exact sequence

$$0 \longrightarrow V \longrightarrow \mathcal{F} \xrightarrow{1-\tau_s} \mathcal{F} \longrightarrow 0 \tag{3}$$

of sheaves on S_{et}, and again there is a long exact sequence

$$\cdots \longrightarrow \mathrm{H}^i(S_{\mathrm{et}}, V) \longrightarrow \mathrm{H}^i(S, \mathcal{F}) \xrightarrow{1-\tau_s} \mathrm{H}^i(S, \mathcal{F}) \longrightarrow \cdots$$

relating the étale cohomology of S with coefficients in V to the Zariski cohomology of the quasi-coherent $\mathcal{O}_S$-module $\mathcal{F}$. This generalizes the long exact sequence (2) to 'twisted' coefficients.

Böckle-Pink and constructible coefficients. A natural problem is now to extend Katz's theorem from locally constant to *constructible* $\mathbf{F}_p$-modules on S_{et}. A strikingly elegant answer was provided by Gebhard Böckle and Richard Pink, in their monograph [**11**]. Let S be a noetherian scheme over $\mathbf{F}_p$. Consider the category $\mathbf{Coh}_\tau\, S$ of pairs $(\mathcal{F}, \tau)$ consisting of a coherent $\mathcal{O}_S$-module $\mathcal{F}$ (not necessarily locally free) and an $\mathcal{O}_S$-linear map $\tau\colon \sigma^\star\mathcal{F} \to \mathcal{F}$ (not necessarily an isomorphism). Such a pair $(\mathcal{F}, \tau)$ defines a constructible $\mathbf{F}_p$-module V on S_{et} by the short exact sequence

$$0 \longrightarrow V \longrightarrow \mathcal{F} \xrightarrow{1-\tau_s} \mathcal{F} \longrightarrow 0.$$

The resulting functor from $\mathbf{Coh}_\tau\, S$ to the category of constructible $\mathbf{F}_p$-modules is not an equivalence. Indeed, if τ_s is nilpotent, then $1 - \tau_s$ will be an isomorphism and hence $(\mathcal{F}, \tau)$ will be mapped to the zero sheaf. Böckle and Pink prove that the full subcategory consisting of pairs $(\mathcal{F}, \tau)$ with τ_s nilpotent is a thick (or 'Serre') subcategory. They define the category $\mathbf{Crys}\, X$ of Crystals on X as the quotient category, and show that $\mathbf{Crys}\, X$ is equivalent with the category of constructible $\mathbf{F}_p$-modules on S_{et}. They moreover construct functors $f^\star$, $f_!$ and $\otimes$ between categories of crystals, compatible with the corresponding functors between categories of constructible $\mathbf{F}_p$-modules.

A different 'quasi-coherent' description of the category of constructible $\mathbf{F}_p$-modules is due to Emerton and Kisin [**19, 20**].

Sheaves and functions. Let X be a scheme of finite type over $\mathbf{F}_q$, and $\mathcal{F}$ a constructible ℓ-adic sheaf on X. For every $x \in X(\mathbf{F}_q)$ the sheaf $x^\star\mathcal{F}$ on $(\operatorname{Spec}\mathbf{F}_q)_{\text{et}}$ is a $\mathbf{Q}_\ell$-vector space equipped with a Frobenius endomorphism Frob. Taking traces, we obtain a function

$$\operatorname{tr}_{\mathcal{F}}\colon X(\mathbf{F}_q) \to \mathbf{Q}_\ell,\ x \mapsto \operatorname{tr}_{\mathbf{Q}_\ell}(\mathrm{Frob} \mid x^\star\mathcal{F}),$$

called the *trace function* of $\mathcal{F}$. The "dictionnaire faisceaux-fonctions" of Grothendieck and Deligne expresses the effect of various functors applied to constructible ℓ-adic sheaves on their trace functions, see [**39**, §1]. We just give two important examples. If $\mathcal{F}$ and $\mathcal{G}$ are constructible ℓ-adic sheaves on X then

$$\operatorname{tr}_{\mathcal{F}\otimes\mathcal{G}}\, x = (\operatorname{tr}_{\mathcal{F}}\, x)\cdot(\operatorname{tr}_{\mathcal{G}}\, x)$$

for all $x \in X(\mathbf{F}_q)$. If $f\colon X \to Y$ is a proper map between schemes of finite type over $\mathbf{F}_q$ then we have

$$\sum_n (-1)^n \operatorname{tr}_{\mathrm{R}^n f_\star\mathcal{F}}\, y = \sum_{\substack{x\in X(\mathbf{F}_q)\\ f(x)=y}} \operatorname{tr}_{\mathcal{F}}\, x$$

by the Lefschetz trace formula, and more generally, for a separated $f\colon X \to Y$, provided one replaces $\mathrm{R}^n f_\star$ by $\mathrm{R}^n f_!$. The dictionary constitutes a powerful tool for proving combinatorial identities between characteristic-zero valued functions on $\mathbf{F}_q$-points of varieties over $\mathbf{F}_q$ using ℓ-adic cohomology.

Deligne has similarly shown that the Lefschetz trace formula also holds for étale cohomology with mod p coefficients [**16**, pp. 125–128]. The proof is based on the exact sequence (3), and on the Woods Hole trace formula in coherent cohomology [**27**, III.6.12]. In fact, the theory of of Böckle and Pink now gives a full "sheaves-functions dictonary", translating between cohomological constructions with coherent sheaves equipped with a Frobenius endomorphism and combinatorics of $\mathbf{F}_q$-valued functions on $\mathbf{F}_q$-points of varieties over $\mathbf{F}_q$. This formalism no longer refers to the étale site, and all statements and proofs can be given in terms of coherent sheaves.

The present lecture notes. The present notes constitute a slightly expanded and more polished version of a series of eight lectures given at the Morningside Center for Mathematics in Beijing in October 2013.

Starting from scratch, we explain the theory of crystals of Böckle and Pink, and how it leads to a sheaves-functions dictionary, translating back and forward between the combinatorics of $\mathbf{F}_q$-valued functions on rational points on varieties over $\mathbf{F}_q$ and the cohomology of coherent sheaves equipped with a Frobenius endomorphism. We illustrate the power of this formalism with a series of applications, ranging from classical results on oscillating sums, and zeta functions modulo p to recent results on special values of characteristic p-valued L-functions.

In *Chapters 1 and 2* we expose part of the theory of crystals of Böckle and Pink. By restricting to $\mathbf{F}_q$-coefficients, by various finiteness assumptions, and by using the theorem of formal functions to give a short new proof of proper base change, we are able to keep the necessary prerequisites to a minimum, and to condense the fundamentals into a relatively concise account.

Chapter 3 contains the central result of these notes: the trace formula for crystals, and the resulting sheaves-functions dictionary. The statement reduces quickly to the special case of projective variety over a finite field. Rather than deducing it from the Woods Hole trace formula of SGA 5, we follow Fulton's very elegant and elementary proof [**22**] to settle this case (filling in a gap in the original argument along the way). By passing to Grothendieck groups of crystals, we both avoid the use of derived categories, and streamline the exposition. *Chapter 4* gives some elementary applications of the trace formula, and *Chapter 5* generalizes the trace formula to crystals with coefficients in various $\mathbf{F}_q$-algebras. Rather than developing the theory with coefficients right from the start we have opted to postpone the introduction of coefficients until Chapter 5, and deducing the general results from their special cases treated in the first three chapters.

We hope our gradual approach in chapters 1–5 will be valued by those who wish to learn to use the sheaves-functions dictionary, but may be intimidated by the large edifice of the full theory of Böckle and Pink.

Chapter 6 computes the cohomology of the "external" symmetric powers of a coherent sheaf on a curve. These are coherent sheaves on the symmetric powers of the curve. In principle this is a special case of a much more general result of Deligne, which expresses the coherent cohomology of symmetric powers on higher-dimensional varieties using simplicial techniques going back to Dold and Puppe. By restricting to dimension 1, we manage to avoid simplicial machinery and obtain a completely explicit statement with a relatively elementary proof. Our proof uses Čech cohomology and Koszul resolutions. The main result in this chapter does not involve Frobenius and holds in arbitrary characteristic. Since it may be of independent interest, care has taken that it can be read independently of the preceding chapters.

In *Chapter 7* we apply the results of Chapter 6 to prove an L-function version of the trace formula of Chapter 5. Since we work with p-torsion coefficients, the charactersitic polynomial of an endomorphism is not determined by the traces of its powers, and we cannot rely on the usual tricks to simply reduce the L-function version to the trace formula for powers of the Frobenius. Rather, we closely mimic Deligne's approach in SGA 4 and SGA 4.5 and use symmetric powers to reduce to the trace formula. A completely different proof is given in Böckle-Pink, based on Serre duality and Anderson's "elementary approach".

We end in *Chapter 8* with an application of the obtained results. We use the main theorem of Chapter 7 to compute special values of L-functions, in particular values of Goss zeta functions at negative integers. The principal result is a generalization of a recent theorem of V. Lafforgue. Under a certain semi-simplicity hypothesis, it expresses special values in terms of extension groups of crystals. It is a characteristic p valued analogue of conjectures and results by K. Kato and Milne and Ramachandran. We end with a simple example showing that the semi-simplicity hypothesis is not always verified. This is contrary to the classical setting of ℓ-adic representations coming from smooth projective varieties over finite fields, where semi-simplicity is conjectured to hold in general.

The appendix gives a self-contained proof of the Woods Hole trace formula for a transversal endomorphism of a proper smooth scheme over a field, using Grothendieck-Serre duality. This is logically independent of the rest of these notes, as these give an independent proof, due to Fulton, for the Frobenius endomorphism. However, since the only published proof of this more general trace formula [**27**, III.6] is rather convoluted, we have decided to include a simpler proof in these notes.

Prerequisites and organization. Although many of the results are closely related to the formalism of étale constructible sheaves, there is no logical dependency, and we do not assume that the reader is familiar with the étale theory. The only prerequisite is familiarity with coherent cohomology at the level of Chapter 3 of Hartshorne [**31**], except for the Leray spectral sequence. We do not make use of derived categories (except for in the appendix, where we need them to state Grothendieck-Serre duality), although throughout the text we retain some of their power and flexibility by an extensive use of Grothendieck groups "$K_0(-)$".

The *Stacks Project* of Johan de Jong and his collaborators [**45**] is rapidly becoming one the most clear, complete and precise references for the foundations of modern algebraic geometry, and we refer to it extensively.

At the end of each chapter are short sections called 'Notes' and 'Exercises'. The former contains historic remarks, comments on some more advanced topics, and references to the literature. In particular, rather than attributing every single lemma and proposition locally, we indicate the origin of the results here. The exercises are of widely varying level of difficulty. Those that require more background are marked with a ($\star$).

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CHAPTER 1

τ-sheaves, crystals, and their trace functions

We fix a finite field $\mathbf{F}_q$ with q elements. Let X be a scheme over $\mathbf{F}_q$. Denote by $\sigma\colon X \to X$ the Frobenius endomorphism which is the identity on the underlying topological space and is given on functions by

$$\mathcal{O}_X(U) \to \mathcal{O}_X(U),\ r \mapsto r^q.$$

It is a morphism of $\mathbf{F}_q$-schemes.

1. Coherent τ-sheaves

Let X be a scheme over $\mathbf{F}_q$.

Definition 1.1. A *τ-sheaf on X* is a pair $(\mathcal{F}, \tau)$ consisting of a quasi-coherent $\mathcal{O}_X$-module $\mathcal{F}$ and a morphism of $\mathcal{O}_X$-modules

$$\tau\colon \sigma^\star\mathcal{F} \to \mathcal{F}.$$

A *morphism of τ-sheaves* $\varphi\colon (\mathcal{F}_1, \tau_1) \to (\mathcal{F}_2, \tau_2)$ is a morphism $\varphi\colon \mathcal{F}_1 \to \mathcal{F}_2$ of $\mathcal{O}_X$-modules such that the square

$$\begin{array}{ccc} \sigma^\star\mathcal{F}_1 & \xrightarrow{\tau_1} & \mathcal{F}_1 \\ \downarrow{\scriptstyle \sigma^\star\varphi} & & \downarrow{\scriptstyle \varphi} \\ \sigma^\star\mathcal{F}_2 & \xrightarrow{\tau_2} & \mathcal{F}_2 \end{array}$$

commutes. The category of τ-sheaves on X is denoted $\mathbf{QCoh}_\tau\, X$.

We will often write $\mathcal{F}$ for the τ-sheaf $(\mathcal{F}, \tau)$, and $\tau_{\mathcal{F}}$ for the map τ.

Let $\mathcal{F}$ and $\mathcal{G}$ be quasi-coherent $\mathcal{O}_X$-modules. Let $\alpha\colon \mathcal{F} \to \mathcal{G}$ be a morphism of sheaves of abelian groups. We say that α is q-linear if $\alpha(rs) = r^q\alpha(s)$ for all local sections r of $\mathcal{O}_X$ and s of $\mathcal{F}$.

Proposition 1.2. *Let $\mathcal{F}$ and $\mathcal{G}$ be quasi-coherent $\mathcal{O}_X$-modules. Then the following sets are in natural bijection:*

(1) $\{\tau\colon \sigma^\star\mathcal{F} \to \mathcal{G} \mid \tau$ *is* $\mathcal{O}_X$*-linear*$\}$,

(2) $\{\tau_a \colon \mathcal{F} \to \sigma_\star \mathcal{G} \mid \tau_a$ *is* $\mathcal{O}_X$*-linear*$\}$,
(3) $\{\tau_s \colon \mathcal{F} \to \mathcal{G} \mid \tau_s$ *is* q*-linear*$\}$.

The subscript a stands for *adjoint*, the s for *semi-linear*.

PROOF. By adjunction we have $\mathrm{Hom}_{\mathcal{O}_X}(\sigma^\star \mathcal{F}, \mathcal{G}) = \mathrm{Hom}_{\mathcal{O}_X}(\mathcal{F}, \sigma_\star \mathcal{G})$. Since σ is the identity on the topological space X we have a canonical isomorphism

$$\alpha \colon \sigma_\star \mathcal{G} \to \mathcal{G}$$

as sheaves of abelian groups. As a map of $\mathcal{O}_X$-modules it is q-linear. The map

$$\tau_s \mapsto \tau_a := \alpha \tau_s$$

gives the bijection between the second and third sets of maps in the proposition. □

DEFINITION 1.3. Assume X is noetherian.* A τ-sheaf $(\mathcal{F}, \tau)$ on X is called *coherent* if the underlying $\mathcal{O}_X$-module $\mathcal{F}$ is coherent. A morphism of coherent τ-sheaves is a morphism of τ-sheaves. The category of coherent τ-sheaves on X is denoted $\mathbf{Coh}_\tau\, X$. If R is an $\mathbf{F}_q$-algebra then we will often write $\mathbf{Coh}_\tau\, R$ in stead of $\mathbf{Coh}_\tau \operatorname{Spec} R$.

EXAMPLE 1.4. Let $X = \operatorname{Spec} R$ for some $\mathbf{F}_q$-algebra R. Let $\mathcal{F}$ be the quasi-coherent $\mathcal{O}_X$-module corresponding to an R-module M. Then $\sigma^\star \mathcal{F}$ corresponds to the R-module

$$R \otimes_{\sigma, R} M,$$

with R-module structure coming from the left factor, and where σ denotes the map $R \to R, r \mapsto r^q$. To give $\mathcal{F}$ the structure of a τ-sheaf is therefore the same as giving an R-linear map

$$\tau \colon R \otimes_{\sigma, R} M \to M.$$

The induced q-linear map τ_s becomes on global sections the map

$$\tau_s \colon M \to M,\ m \mapsto \tau(1 \otimes m)$$

satisfying $\tau_s(rm) = r^q \tau_s(m)$. Conversely, any such map determines a map $\mathcal{F} \to \sigma_\star \mathcal{F}$ of quasi-coherent $\mathcal{O}_X$-modules, and therefore the structure of a τ-sheaf on $\mathcal{F}$.

If R is noetherian then the τ-sheaf $\mathcal{F}$ is coherent if and only if M is a finitely generated R-module.

*For simplicity, we will restrict ourselves to noetherian schemes whenever dealing with *coherent* $\mathcal{O}_X$-modules. With the necessary care many of the results in this text could be extended to cover τ-sheaves over more general schemes.

Let $f\colon X \to Y$ be a morphism of schemes over $\mathbf{F}_q$. Then $\sigma_Y \circ f = f \circ \sigma_X$. In particular, for a quasi-coherent $\mathcal{O}_Y$-module $\mathcal{F}$ we have a canonical isomorphism

$$\sigma_X^\star f^\star \mathcal{F} \longrightarrow f^\star \sigma_Y^\star \mathcal{F}.$$

Definition 1.5. Let $\mathcal{F} = (\mathcal{F}, \tau)$ be a τ-sheaf on Y. The *pullback* or *base change* $f^\star\mathcal{F}$ of $\mathcal{F}$ along f is the τ-sheaf $(f^\star\mathcal{F}, \tau_{f^\star\mathcal{F}})$, where $\tau_{f^\star\mathcal{F}}$ is the composition

$$\begin{tikzcd} \sigma_X^\star f^\star \mathcal{F} \arrow[r, "\sim"] \arrow[dr, "\tau_{f^\star\mathcal{F}}"'] & f^\star \sigma_Y^\star \mathcal{F} \arrow[d] \\ & f^\star\mathcal{F}. \end{tikzcd}$$

Pullback defines a functor $f^\star\colon \mathbf{QCoh}_\tau\, Y \to \mathbf{QCoh}_\tau\, X$.

Example 1.6. Assume $Y = \operatorname{Spec} R$ and $X = \operatorname{Spec} S$ and $f\colon X \to Y$ induced by an $\mathbf{F}_q$-algebra homomorphism $R \to S$. Assume that $\mathcal{F}$ corresponds to an S-module M equipped with a q-linear $\tau_s\colon M \to M$. The map

$$\tau_s'\colon S \otimes_R M \to S \otimes_R M,\ s \otimes m \mapsto s^q \otimes \tau_s(m)$$

is well-defined and q-linear. The pair $(S \otimes_R M,\, \tau_s)$ corresponds to the pull-back $f^\star\mathcal{F}$.

Proposition 1.7. *Let $f\colon X \to Y$ be a morphism of noetherian schemes over $\mathbf{F}_q$. Let $\mathcal{F}$ be a coherent τ-sheaf on Y. Then $f^\star\mathcal{F}$ is a coherent τ-sheaf.*

Proof. Since X is noetherian, $\mathcal{O}_X$ is a coherent $\mathcal{O}_X$-module [**28**, I.1.6.1] [**45**, Tag 01XY], and therefore the pull-back of any coherent $\mathcal{O}_Y$-module is a coherent $\mathcal{O}_X$-module [**28**, 0.5.3.11] [**45**, Tag 01BM]. □

Proposition 1.8. *Let X be a scheme over $\mathbf{F}_q$. The category $\mathbf{QCoh}_\tau\, X$ is abelian. If X is noetherian then also $\mathbf{Coh}_\tau\, X$ is abelian.*

Proof. Clearly the categories are additive. We need to show that they satisfy

(AB1) Every morphism $\varphi\colon \mathcal{F} \to \mathcal{G}$ has a kernel and cokernel,
(AB2) For every morphism $\varphi\colon \mathcal{F} \to \mathcal{G}$ the natural map

$$\operatorname{coker}(\ker\varphi \to \mathcal{F}) \to \ker(\mathcal{G} \to \operatorname{coker}\varphi)$$

is an isomorphism.

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Since σ is the identity on the underlying topological space of X the functor $\sigma_\star$ on quasi-coherent $\mathcal{O}_X$-modules is exact. In particular, a morphism $\varphi\colon \mathcal{F} \to \mathcal{G}$ of quasi-coherent τ-sheaves induces a commutative diagram of $\mathcal{O}_X$-modules with exact rows

$$\begin{tikzcd}
0 \arrow[r] & \ker\varphi \arrow[r] \arrow[d, "\tau_a"] & \mathcal{F} \arrow[r, "\varphi"] \arrow[d, "\tau_a"] & \mathcal{G} \arrow[r] \arrow[d, "\tau_a"] & \operatorname{coker}\varphi \arrow[r] \arrow[d, "\tau_a"] & 0 \\
0 \arrow[r] & \sigma_\star \ker\varphi \arrow[r] & \sigma_\star\mathcal{F} \arrow[r, "\sigma_\star\varphi"] & \sigma_\star\mathcal{G} \arrow[r] & \sigma_\star \operatorname{coker}\varphi \arrow[r] & 0
\end{tikzcd}$$

One directly verifies that $(\ker\varphi, \tau_a)$ and $(\operatorname{coker}\varphi, \tau_a)$ determine a kernel respectively cokernel of the morphism φ in $\mathbf{QCoh}_\tau\, X$. Property AB2 is inherited by the same property for the category of quasi-coherent $\mathcal{O}_X$-modules, hence $\mathbf{QCoh}_\tau\, X$ is is abelian.

The coherent $\mathcal{O}_X$-modules form an abelian subcategory of the category of quasi-coherent $\mathcal{O}_X$-modules and the same arguments as above show that $\mathbf{Coh}_\tau\, X$ satisfies AB1 and AB2. □

In the proof we have used the adjoint maps τ_a to produce kernels and cokernels. The main advantage is that $\sigma_\star$ is an exact functor. In general the functor $\sigma^\star$ is not exact† and it takes a bit more work to construct kernels and cokernels directly in terms of the maps $\sigma^\star\mathcal{F} \to \mathcal{F}$. Let us, as an example, describe in detail the kernel of a map $\varphi\colon (\mathcal{F}, \tau_{\mathcal{F}}) \to (\mathcal{G}, \tau_{\mathcal{G}})$ of τ-sheaves on X. Let $\mathcal{H}$ be the $\mathcal{O}_X$-module which is the kernel of $\varphi\colon \mathcal{F} \to \mathcal{G}$. Consider the commutative diagram of $\mathcal{O}_X$-modules

$$\begin{tikzcd}
 & \sigma^\star\mathcal{H} \arrow[r] \arrow[d, dashed, "\tau_{\mathcal{H}}"'] \arrow[dr] & \sigma^\star\mathcal{F} \arrow[r, "\sigma^\star\varphi"] \arrow[d, "\tau_{\mathcal{F}}"] & \sigma^\star\mathcal{G} \arrow[d, "\tau_{\mathcal{G}}"] \\
0 \arrow[r] & \mathcal{H} \arrow[r] & \mathcal{F} \arrow[r, "\varphi"] & \mathcal{G},
\end{tikzcd}$$

where the bottom row is exact. The map $\sigma^\star\mathcal{H} \to \sigma^\star\mathcal{G}$ is the pullback along σ of the map $\mathcal{H} \to \mathcal{G}$, and hence it is the zero map. It follows that the map $\sigma^\star\mathcal{H} \to \mathcal{F}$ factors over a unique map $\tau_{\mathcal{H}}\colon \sigma^\star\mathcal{H} \to \mathcal{H}$. The pair $(\mathcal{H}, \tau_{\mathcal{H}})$ is the kernel of φ in $\mathbf{QCoh}_\tau\, X$. If $\mathcal{F}$ and $\mathcal{G}$ are coherent then so is $\mathcal{H}$ and then $(\mathcal{H}, \tau_{\mathcal{H}})$ is also the kernel of φ in $\mathbf{Coh}_\tau\, X$.

†For example, if $X = \operatorname{Spec} R$ for a noetherian local ring R then the functor $\sigma^\star$ on quasi-coherent $\mathcal{O}_X$-modules is exact if and only if R is regular, see [**37**]. See also Proposition 8.9.