# PARADOXES AND INCONSISTENT MATHEMATICS

Logical paradoxes – like the Liar, Russell's, and the Sorites – are notorious. But in *Paradoxes and Inconsistent Mathematics*, it is argued that they are only the noisiest of many. Contradictions arise every day, from the smallest points to the widest boundaries. In this book, Zach Weber uses *dialetheic paraconsistency* – a formal framework where some contradictions can be true without absurdity – as the basis for developing this idea rigorously, from mathematical foundations up. In doing so, Weber directly addresses a longstanding open question: how much standard mathematics can paraconsistency capture? The guiding focus is on a more basic question, of why there are paradoxes. Details underscore a simple philosophical claim: that paradoxes are found in the ordinary, and that is what makes them so extraordinary.

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A paradox
is only the truth
standing
on
its
head to get attention.

— Oscar Wilde

# Contents

# Preface

*Why are there paradoxes?* That there *are* logical paradoxes is not in question. A notorious bunch of them besets core parts of our understanding the world:

- The liar paradox in theories of truth
- Russell's paradox in theories of sets and properties
- The sorites paradox in any theory with vagueness

Many proposals have been offered that address aspects of these problems, from seminal work [Russell, 1905b; Tarski, 1944; Kripke, 1975] to the more recent [Priest, 2006b; Field, 2008; Beall, 2009]. Modern logic has given us unprecedented insight into the mechanics, so to speak, of paradoxes – we know *how* they arise, and how, at least temporarily, they can be evaded. The question investigated in this book is more basic: asking for an explanation of *why* there are paradoxes at all.

Paradoxes can look like they are unusual accidents, exceptional borderline cases, self-referential anomalies at the edge of the world. And paradoxes certainly are found at these dramatic and distant limits – but not only there. The famous paradoxes are only the noisiest of many. In the pages ahead, I rethink the paradoxes as much smaller in scale, appearing everywhere as innocuous parts of everyday objects. They are found at the edge of the universe but also at the edge of a coffee cup. Shifting our thinking to the local level demystifies the problem. Rather than fixating on bizarre things in bizarre places, we can begin to appreciate that paradoxes are found in the ordinary: *that* is what makes them so extraordinary.

The paradoxes are deductive arguments that end in contradictions; they appear to be *proofs* of contradictions. According to prevailing views, though, a contradiction can only be a mistake. So according to prevailing views, the paradoxes simply cannot be what they appear to be. But this classical approach rules out a priori the simplest explanation of the paradoxes: that *they are exactly what they appear to be*.

The framework in this book, then, uses *paraconsistent logic*, in the *dialetheic* tradition: the framework that allows for some contradictions to be true, without everything whatsoever being true.[1] New understanding of the paradoxes becomes possible – and precise – by situating them where they first arise, in foundational logico-mathematics, but now described with formalism specially designed to handle inconsistency. Central chapters develop elementary axiomatic theories in paraconsistent logic, beginning with Frege–Cantor naive set theory, and going on to describe rudiments of arithmetic, algebra, real analysis, and topology, with a focus throughout on paradoxes as they arise at *boundaries*. In doing so, this work thus addresses a longstanding open question, sometimes phrased as an objection: "Can paraconsistency capture any standard mathematics? If so, how much?" The (qualified) answer provided is by direct demonstration.

Crucially, I argue that the entire presentation ought to be purely paraconsistent, from the object level to the "meta"-level, without any recourse to classical resources (including right now). What is the picture of the world that emerges when we describe it in a fully nonclassical language? I provide a sketch, emphasizing qualitative aspects that localize the paradoxes at geometric *points*. With an appropriate paraconsistent framework, we can understand paradoxes as bona fide mathematical and metaphysical objects–explained, rather than explained away.

* * *

Here is the **argument in a nutshell**:

I There are true contradictions, both in the foundations of logic and mathematics, and in the everyday world.[2]

II If the world is inconsistent but not absurd, then the logic underlying our theory of the world, including all of logic and mathematics, ought to be paraconsistent.

III Paraconsistent logic then must, and can, show that it supports some ordinary reasoning. A minimum requirement is that the logic be able to reestablish the motivating paradoxes – *proving* the contradictions, on their own terms, in elementary mathematics.

IV In proving the paradoxes paraconsistently, the basic components of a nonclassical picture come into view (including nonstandard descriptions of identity, boundaries, and points). Then we are finally positioned to (re)address the question of why there are paradoxes.

For (I), I largely follow standard arguments for dialetheism, and in this obviously owe an enormous debt to Graham Priest, Richard Routley/Sylvan, and other pioneers[3] in

[1] The term "paraconsistent" was invented by Miró Quesada in the mid-1970s. The term "dialetheic" was invented in the late 1970s by Graham Priest and Richard Sylvan (Routley at the time) [Priest et al., 1989, p. xx], to replace the term "dialectic," which had been in use from Hegel. Dialetheism is simply the thesis that there are true contradictions; etymologically, the word is intended to evoke something like "two-way truth," and is sometimes spelled "dialethism." The (somewhat) less esoteric term "glut theory" means the same thing, and is used too, e.g., by Beall [Beall, 2009].

[2] *Being contradictory* is a property of sentences (or propositions), so literally speaking, a coffee cup can't be contradictory or "have a contradiction on it"; but if there is a *true description* of the coffee cup that is contradictory, then I will say that the coffee cup is too. Here and throughout, to say that *the world is inconsistent*, or that there are true contradictions in the world, is an evocative shorthand to mean that some glutty theory of the world is true (but without thereby committing some elementary category mistake). See Section 0.2.2 and [Priest, 2006b, pp. 299–302].

[3] Some key works by Priest and Roultey are [Routley, 1977; Priest and Routley, 1983; Priest, 2006b]. Other important contributors (in varying ways) include Arruda [Arruda and Batens, 1982], Asenjo [Asenjo, 1966, 1975], Beall [Beall, 2009], Brady [Brady, 2006], Colyvan [Colyvan, 2008a], da Costa [da Costa, 1974], Dunn [Dunn, 1980], Meyer [Meyer, 1976], Mortensen [Mortensen, 1995], Restall [Restall, 1992], Slaney [Slaney, 1982], and others.

paraconsistency and *inconsistent mathematics*: the application of paraconsistent logic to the study of contradictory abstract structures. In the Introduction and Chapter 1, I will offer various examples and considerations that I think motivate dialetheism; I don't simply assume that there are true contradictions, and I am almost certain that you don't. (Please write to me if you do.) Nevertheless, while the thesis that there are truth gluts remains contentious, I think to advance from here does not require further rehearsal of abstract polemics about the law of noncontradiction or the like. I don't try to defend my approach point-for-point against other options, because what is most important for the project now is to show what it can *do*. Dialetheic paraconsistency is a *motive* and a *method*; I argue that it both independently supports some genuine mathematical reasoning and, in doing so, offers new insights into the paradoxes. My plan is mainly just to show what a committed paraconsistent picture of the world drawn with precise tools might look like, taking a "deep dive" into the substructure of inconsistent space.

So in taking proposition (I) to be largely defended elsewhere, it is in the scope and depth of propositions (II) and (III) – the question of how mathematically revisionary dialetheic paraconsistency might be, and the "classical recapture" – that I am predominantly attempting new advances, with (IV) a matter of setting targets – a "paradox recapture" – and following out our commitments to their logico-mathematical end. I am sketching, to put it colorfully, a possible world in which the reaction to the paradoxes circa 1900 was quite different, and subsequently the mathematics that developed in the twentieth century had a different tenor, preserving obvious and intuitive principles (such as "collections are sets") by treading more carefully with our logic. This is not *Principia Mathematica Paraconsistenta* – the mathematical chapters are not strictly cumulative – but perhaps indicates how such a tome could eventually (or why it never will) be written. This is not a textbook. I hope it might invite others to use it as a stepping stone.

If some of the claims in this book seem extreme (the word "unhinged" may cross your mind – it has mine), they are nevertheless motivated by very traditional, almost Socratic sorts of commitments: the world can be made sense of, even (especially) when it seems senseless. Or at least, it is incumbent upon us to try. The "argument in a nutshell" has an unstated Premise 0: the world is ultimately and profoundly *intelligible*, in something like the mode of the Enlightenment project and the principle of sufficient reason, up to Hilbert's program.

> Is this axiom of the solvability of every problem . . . a general law inherent in the nature of mind, that all questions which it asks must be answerable? . . . We hear within us the perpetual call: There is the problem. Seek the solution. You can find it by pure reason, for . . . there is no *ignorabimus* . . . *[Hilbert, 1902b]*

The spirit of this work is with rationalists and button-down logicists, give or take a disjunctive syllogism or two. I think that the challenge posed by the paradoxes is a challenge to reason itself, and that we are called to respond: that even (especially) when reason gives out, that is exactly when not to abandon reason. I ultimately believe that the paradoxes exist because *the world has no gaps*, that as Leibniz said, "*La nature ne fait jamais de sauts*."[4]

[4] E.g., in the Preface to the *New Essays*, [Leibniz, 1951, p. 378].

(If the world has gaps, those gaps are part of the all-inclusive world, too.) I am transfixed by Leibniz's beautiful statement from the *Monadology* [Leibniz, 1714, §47] that all objects are

> generated by the continual flashes of silent lightning [*fulgurations continuelles*] ...

By the end, I hope to have discovered a little of what this could mean.

In 1931, Gödel famously dashed Hilbert's hopes, if not the hopes of the entire Enlightenment project, by using a version of the liar paradox to prove that any complete, axiomatic theory of the world will be inconsistent. I read this result as an invitation. Welcome to our inconsistent world.

* * *

**Acknowledgments**

This work draws on papers published over my career to date. In putting this all together, everything is either extensively rewritten, remixed, or entirely new; but descendants of previously published work (or, published work that descended from this book) include the following. Chapter 1§2 draws from [Weber, 2010c], and 1§3 from [Weber and Cotnoir, 2015]. Chapter 2 and bits of Chapter 3 are modified from [Weber, 2019]. Some of Chapter 3§1 and 3§2 draws from from [Weber et al., 2016]. Some of Chapter 4 began in [Badia and Weber, 2019]. Thank you to my coauthors, and thank you to the original publishers for permission to draw on these articles. Additionally, sections throughout owe debts to (at least) the following: [Weber, 2009; 2010d; Weber and Colyvan, 2010; Weber, 2010d; McKubre-Jordens and Weber, 2012; Caret and Weber, 2015; Meadows and Weber, 2016; Girard and Weber, 2019; Omori and Weber, 2019].

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