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Excerpt
More Information

Part I

What Are the Paradoxes?

> *There are whole mathematical cities that have been closed off and partially abandoned because of the outbreak of isolated contradictions ...*
>
> *They have become like modern restorations of ancient cities, mostly just patched up ruins visited by tourists* [Routley, 1977, p. 927].

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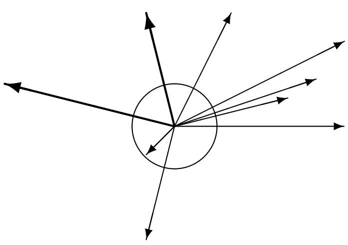
Introduction to an Inconsistent World



Look! The sun is rising on the horizon, where the earth meets the sky. The earth is not the sky and the sky is not the earth, but they touch, and these together are the world. The horizon: sky and not sky, earth and not earth, while the dawn chorus of birds sings. It is just an ordinary day, with paradoxes right in front of you.

0.1 The Problem

0.1.1 There Are Contradictions Inside Truisms

A *paradox* is a seemingly sound argument to a seemingly false conclusion.[1] A paradox is *genuine* when things are as they seem: a genuine paradox *is* a valid argument with true premises and a false conclusion. Since a valid argument preserves truth from premises to conclusion, the conclusion of a paradox is also true. The most striking paradoxes present themselves as logical-mathematical *proofs* of propositions that are both true and false. The conclusion of a genuine paradox, if there is any such thing, is a *dialetheia*, or *glut*, a true contradiction.

The paradoxes I am concerned with are very easy to state. Their contradictory conclusions are derived with a minimum of logical resources. They follow from principles that seem like they could not fail to be true, things we know to a certainty:

- A proposition p is true iff it is the case that p.
- An object a is in the set of φs iff it is the case that a is φ.
- Sometimes it is raining, and sometimes it is not.

[1] Following standard usage, following [Sainsbury, 1995], following Quine; see [Lycan, 2010].

And yet these ineluctable truths, these banalities, enclose inconsistencies – respectively, the liar paradox, the Russell paradox, and the sorites paradox.[2] Concentrated efforts to make it otherwise have foundered, despite their ingenuity and sophistication. This is fascinating: *vast* energies have been expended on showing that the paradoxes are not genuine, that things *could not be* as they seem. Through it the paradoxes have remained; they have *flourished*.

One good indicator of truth is that it persists, especially through attempts to repress or deny it. And so a good indicator of a genuine paradox is that it is somehow irrepressible – or as we have learned to say, it exhibits *revenge*.[3] Upon solution of a paradox, a new paradox arises that is essentially the same as the original paradox. To use a kind of geometric metaphor, the genuine paradoxes look to be *invariants* of certain spaces under solution-transformations. They do not go away.

This is not a pessimistic induction on the failure of logicians to solve some problem. It is a considered reevaluation of the paradoxes as having been unsolvable *for very good reason*: there is nothing to "solve." The idea is that logicians have been like apocryphal Pythagoreans attempting to "solve" the existence of irrational magnitudes such as $\sqrt{2}$ (cf. Chapter 6), whereas the better route forward is to see the rational numbers as only some among many.[4]

Paradoxes seem like a trick, except you can't figure out how the illusion works. And despite your persistence, you *never* figure it out. Every time you think you've got it, explained the deception, the magician repeats the trick, now with your attempted explanation in plain sight. "You see? No self-reference up my sleeve." After long enough, with successively more prolix attempts to expose the trick failing one by one, it becomes more reasonable to consider whether it is somehow not a trick at all.[5] The most amazement a magician can generate, after all, is for the viewer to realize that it isn't an illusion: the magician has done the impossible for real.[6]

Here's the trick. You were once a baby. Now you are not a baby. Because of the nature of time, there must have been a last moment you were a baby, or a first moment you were not. If a change occurred – and it did – then it must have occurred at some point, some *instant*. Even if the change was gradual, then the *beginning* of the gradual change itself still must have been at some precise moment. Or the beginning of the beginning . . . It has to be! But as anyone can tell you, there is no one exact instant when a baby stops being a baby. So it looks like you changed in a profound way, without the change occurring anywhere. It's like escaping from a locked box without ever passing through its surfaces. Teleportation!

This problem – the *sorites paradox* – is so hard that it has led some philosophers to deny the existence of babies,[7] among other solutions. But perhaps the vagaries of terms such

2 For *Curry's paradox*, see Chapter 4.
3 Much more about revenge to come. See the introduction of [Beall, 2007].
4 The canonical presentation of the argument I've just sketched is in [Priest, 1979].
5 Mates similarly cautions the reader not to be distracted while "the rabbit of paradox is being brought out of the hat" [Mates, 1981, p. 5]. He says the paradoxes are "both intelligible and insoluble."
6 My favorite example is the "magic trick" of being impaled by a sword, or stabbed with an ice pick, without any pain or blood. This is done by the magician having a special kind of scar tissue in the relevant place – a *fistula*, like an earring hole. So the magician really does just stick a sharp object into themself; that's it. Cf. "Miracle Man," *Time*, June 23, 1947.
7 *Mereological nihilists* say that tables and trees don't exist, though they disagree about living people. See [Unger, 1979; van Inwagen, 1990].

as "is a baby" and assumptions about the nature of time come with enough doubt that you think this oddity can be safely ignored (at least until Section 1.2). After all, a little reflection will show that our days are filled will vagueness, from the dawn (when exactly is it?) to rainstorms (how may drops does it take to be raining?) onward. And how could something so ubiquitous be paradoxical?

So here is the trick again, starting with something that must be true, could not be more certain – Aristotle's definition of truth (enjambment added):[8]

> To say of what is that it is not, or of what is not that it is,
> is false,
> while to say of what is that it is, and of what is not that it is not,
> is true.

A *truth predicate* $\mathsf{T}(x)$ takes a name for any sentence and is satisfied depending on whether or not that sentence holds, whether the state of affairs described by that sentence is the case. It says of what is, that it is, and of what is not, that it is not. In 1936, Tarski put a precise schematic form around this: with arrows for implication, for any sentence φ,

Truth schema $\mathsf{T}(\ulcorner\varphi\urcorner) \leftrightarrow \varphi$,

where $\ulcorner\cdot\urcorner$ is a name-forming operator on sentences. It is true that φ if and only if φ: this is the schema for *naive truth theory.*

Why "naive"? Because there is a sentence ℓ called "the liar" that says of itself that is it false; with $\neg$ representing negation:

$$\ell \leftrightarrow \neg\mathsf{T}\ulcorner\ell\urcorner. \tag{0.1}$$

Sentence ℓ says that sentence ℓ is not true. Putting (0.1) together with the instance of the truth schema $\mathsf{T}\ulcorner\ell\urcorner \leftrightarrow \ell$, we have

$$\mathsf{T}\ulcorner\ell\urcorner \leftrightarrow \neg\mathsf{T}\ulcorner\ell\urcorner \tag{0.2}$$

using the transitivity of biconditionals (if $p \leftrightarrow q$, and $q \leftrightarrow r$, then $p \leftrightarrow r$). Then we reason as follows. Either ℓ is true, or it is not. If ℓ is true, $\mathsf{T}\ulcorner\ell\urcorner$, then $\neg\mathsf{T}\ulcorner\ell\urcorner$, by (0.2) and modus ponens (if p and $p \rightarrow q$, then q). Thus by reductio (if $p \rightarrow \neg p$ then $\neg p$), ℓ is not true,

$$\neg\mathsf{T}\ulcorner\ell\urcorner.$$

But if ℓ is not true, then that is just what ℓ says (!); going through (0.2) again,

$$\mathsf{T}\ulcorner\ell\urcorner$$

after all. So we have established

$$\mathsf{T}\ulcorner\ell\urcorner \;\&\; \neg\mathsf{T}\ulcorner\ell\urcorner. \tag{0.3}$$

Contradiction. And then, assuming that the truth schema is contraposable (contraposition is if $(p \rightarrow q)$ then $(\neg q \rightarrow \neg p)$), then

$$\ell \;\&\; \neg\ell. \tag{0.4}$$

[8] From *Metaphysics* 4.1011b25, echoed in Plato's *Cratylus* 385b2 and *Sophist* 263b.

Thus we appear to have proved both a sentence and its negation using an obviously true axiom (scheme) and elementary propositional logic.[9]

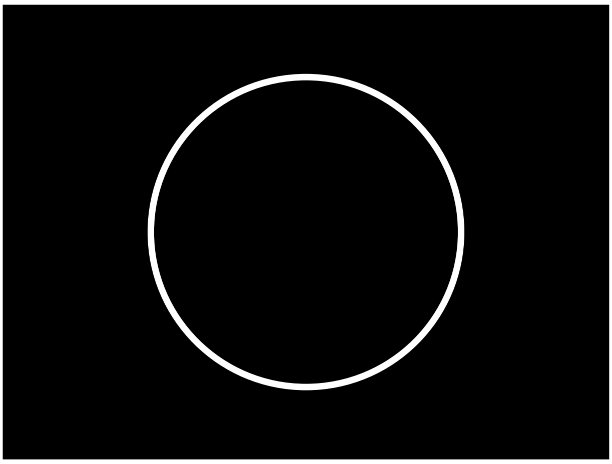
The paradoxes are no trick.[10] Listen to Tarski:

> In my judgment, it would be quite wrong and dangerous from the standpoint of scientific progress to depreciate the importance of this and other antinomies, and to treat them as jokes or sophistries. It is a fact that *we are here in the presence of an absurdity*, that we have been compelled to assert a false statement . . . *[Tarski, 1944, emphasis added]*

The paradoxes are there in the most basic places: sets; truth; raindrops. If there were no genuine paradoxes, only apparent ones, then there would still be a psychosociological project of explaining why so many people have found the paradoxes compelling. But after a while, there being a sufficient number of lucid expert witnesses should start to suggest that maybe they have all witnessed something real. Consider, then, one very gentle and elegant move. The paradoxes look unavoidable, cannot be eliminated, because they are "what they always seemed to be, proofs" [Routley, 1979, p. 302]. The conclusions of the paradoxical arguments are *true*.

0.1.2 There Are Contradictions in Plain Sight in Space

Paradoxes are not confined to abstruse contemplation of self-referring sentences or vague predicates. They have a simple visual presentation. Here is a circle:



It divides the plane in which it sits into an interior and an exterior. There is a continuous path between any two points in the exterior, and any two points in the interior. But a path from a point in the exterior will not be able to reach a point in the interior without crossing the circle, which forms the *boundary* of the two parts. It looks like the plane is divided

9 See Section 1.1.2, [Priest, 2006b, ch. 1], and book-length treatment in [Beall, 2009].

10 "There are scarcely any philosophical problems of greater urgency than the liar paradox, for there are scarcely any concepts more central to our philosophical understanding than the concept of truth. . . . Quite unmistakably, our present way of thinking about truth and reference is inconsistent" [McGee, 1990, p. vii].

exclusively and *exhaustively*: assuming continuity of the plane, every point is either in the interior or the exterior, and not both. Except – what about the points on the boundary itself?

An even simpler version of this problem is in one dimension. Given a continuous line segment,

it is obvious that the line can be perfectly divided into, or is composed of, two perfectly symmetrical halves.[11] But, as your idealized knife comes down to make the cut, it touches the *center point* of the line. There must be a point there, because the line has no gaps. But points themselves have no extension and so cannot be divided; so your knife must slip either to the left or to the right of the center point. Then the resulting two pieces are not perfectly even: one piece has the center point, and the other does not.

Dividing anything in two – e.g. the left side of a line and the not-left side, the truths and the falsities, the babies and the non-babies – calls attention to the logical assumption of bivalence: that every sentence is either true or false. The implications of this assumption have been well appreciated at the "cosmic" level. According to Gödel, paradoxes are in fact due to the (purportedly mistaken) notion of

> dividing the totality of all existing things into two categories. *[Gödel, 1964, p. 519]*

But grand talk about dividing the totality of all existing things,[12] the *universe*, distracts from the fact that we have the very same sort of problem, not at the level of the universe, but at *any* medium-sized object. We do not need special properties of connectedness or closed curves to appreciate a problem here; we just need to look at the sun in the sky (or the moon, since you shouldn't stare at the sun) and wonder how it appears to be a distinct object that is nevertheless smoothly embedded in phenomenal space.

The problem localizes in asking, which portion of reality are your hands, and which portion not? The microscopic particles of matter grazing the surface between your skin and the air are a *question*. There are few things more certain than holding up a hand, and saying "here is a hand," as Moore observed.[13] At the same time, "there are always outlying particles, questionably parts of the thing, not definitely included and not definitely not included" [Lewis, 1999, p. 165]. What looks like a special problem for the universe is a commonplace problem occurring on the end of your arm.

It *seems* like a circle can sit in the plane. It *seems* like there is a universe, in which some things are hands and everything else is not. And – here's the pointy end of the stick – it seems like these things must be able to be true without plunging us into abyssal and vexatious mysteries.

Things are not always as they seem.

[11] Gödel once remarked (in an unpublished note; see [Putnam, 1994]) that if a geometric line segment is divided evenly at a point, it would be natural to expect the two halves of the line to be perfectly symmetric mirror images. See Section 9.2.1.

[12] Petersen echoes Gödel's point [Petersen, 2000, p. 384]: "The point is a highly metaphysical one: is it possible, in principle, to divide the world (or the universe, if you prefer) into two disjunct parts, the union of which is the world?"

[13] "Proof of an External World" [1939] in [Moore, 1993, pp. 147–170].

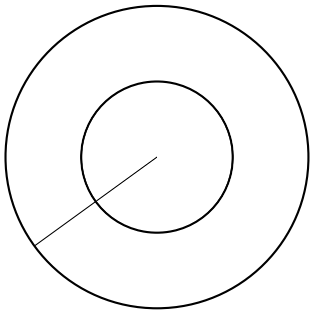
0.1.3 Paradoxes Are Resolved by Reversing the Order of Explanation

Sometimes things are as they seem. Usually, even: accepting things as they are, mostly, is a necessary assumption for getting around in the world. We trust perception, testimony, and the conclusions of informal reasoning as overwhelmingly veridical. Exceptions are exceptional, surprising, unsettling. *Exceptio probat regulam in casibus non exceptis*, my mother always said.

If the first phase of philosophy is to make what is unproblematic into a problem (Cartesian doubt!), then the next phase is to make what is problematic into what is not, to "show the fly the way out of the bottle,"[14] so to speak. There is a place for scientific surprises, but there is also a place for theories that make it possible to understand the world as we find it, theories that do not propound a long story about why everything we seem to see and think is wrong. An account of reality should be available of the reality we live in.[15] Dealing with the paradoxes, it has always seemed to me, should not require an elaborate apology for why the world is not really as it seems.

An impressive practitioner of this method is Richard Dedekind. In 1872, Dedekind famously advanced on the problem of characterizing continuity. In thinking about a geometric line, Dedekind took it as an adequacy condition that the line have no gaps. That is the linear continuum as we find it. And so he proposed, in essence, that *any gap in the line be thought of as itself a point* – indeed that the line is entirely made of "cuts," pairs of sets comprised of everything to the left and everything to the right. Any possible *counter*example is reconsidered as a natural *example*.

Or again, in a magnificent 1888 treatise, Dedekind faced a problem known since Proculus in the third century, that given two concentric circles, a radius cuts each circumference exactly once:



14 [Wittgenstein, 1953, §309].

15 "... the world as I found it" [Wittgenstein, 1922, §5.631]. This methodology is also found in Husserl and the phenomenological tradition – placing focus on the world of appearance and experience. See Tieszen, 2005, ch. 2, 3; [van Atten, 2007].

Apparently, then, there are exactly as many points on both circumferences, since the radius establishes a one-to-one pairing of the points on the inner circle and the points on the outer circle. But, even if the set of points on both circles is infinite – *obviously* the outer circle is bigger, so its circumference has more points!? Dedekind's move was to treat this anomaly as a *definition*: a set is infinite if and only if "it is similar to a proper part of itself" [Dedekind, 1901, def 64]. He didn't deny the data, or try to "solve" the problem, or fall back on quietism or eliminativism about circles. The anomaly that infinity is "bigger than itself" (captured in the simple $\infty + 1 = \infty$) is what makes infinity infinite. The puzzle is the answer. He took his task to be to describe the world as we find it – to explain, rather than explain away.

"What we see of the things are the things," writes Pessoa. "Why would seeing and hearing be to delude ourselves / when seeing and hearing are seeing and hearing?" The world I live in has days and nights and forests and cities, flocks of birds and bundles of recycling. There are words for these things, names and predicates, but not *only* words; the world has the things the words name, too. So there are sets, and there are properties, and there are boundaries, and some are vague and some are inconsistent. The *ordinary* leads us to an impasse. Dedekind shows how an impasse can be turned in to a way out.

0.2 The Choices

Given a paradox, there are only three options:[16]

- Reject the reasoning of the argument as invalid
- Show one of the premises is false
- Accept that the conclusion is true

A brief and biased review of how these options play out, say in the case of the liar (Section 0.1.1), is in order. (It would mostly carry over for the paradoxes of set theory, as per the next chapter, too.) This is well-worn ground, and I only intend a thumbnail sketch of some vast and important literature; various distinctions, objections, and replies are omitted.[17]

On some very familiar assumptions, truth is what is the case, every proposition is either true or false, and no proposition is both true and false. There must be a set of *all and only* the true propositions, divided out perfectly from the falsities. This is to be an *exclusive* and *exhaustive* division. Except, again – what about propositions on the "boundary"? This is the liar.

[16] Of course, not only three. You could excuse yourself and go for a walk. Or in a more philosophical vein, you could try to undercut the trilemma as somehow misconceived or based on some bad presupposition, like "the paradoxical sentences are meaningful." I will assume that the paradoxes are meaningful problems demanding a direct response; and so in that sense, these other options are not options.

[17] More details are in [Priest, 2006b, ch. 1]; see also opening chapters in [Field, 2008; Scharp, 2013], and many other sources, e.g., [Beall et al., 2018].

0.2.1 Incompleteness

0.2.1.1 The "Classical" Solution

Many different sorts of proposals fall under the heading of "classical" solutions to the liar paradox, but most of them share some key features, around the strategy of imposing (or discovering) that truth is somehow indexed, or stratified, structured in a *hierarchy*.

Classical theories deny, or restrict, the completely unrestricted truth schema $\mathsf{T}(\ulcorner\varphi\urcorner) \leftrightarrow \varphi$. This is closely related to Tarski's 1936 theorem, that no (consistent) language can contain its own truth predicate, or be *semantically closed*, on pain of the liar. To spell this out, Tarski gives a construction with an infinite hierarchy of *metalanguages*,[18]

$$\mathcal{L}_0, \mathcal{L}_1, \mathcal{L}_2, \ldots$$

Each language $\mathcal{L}_{n+1}$ at level $n+1$ can look back at (all) the previous one(s) $\mathcal{L}_n$, but not itself; truth claims are only made about "earlier" sentences, so if φ is a sentence of language $\mathcal{L}_n$, then $\mathsf{T}\ulcorner\varphi\urcorner$ is a sentence of $\mathcal{L}_{n+1}$. Self-referential sentences involving the truth predicate are impossible, and so the liar sentence is never formed. The truth schema can hold, but only over a restricted language.[19]

The problems for this approach are basic.[20] First, natural languages *do* contain their own truth predicate. Not only is there no evidence for the existence of any "metalanguages" in natural language, a priori, there cannot be any language that is "beyond" [$\mu\epsilon\tau\alpha$] language. To his credit, Tarski is explicit on this point: natural language *is* semantically closed, and therefore, in his view, mathematically intractable; he abandons trying to analyze the full concept of truth. Later approaches, such as [Kripke, 1975], would purport to be otherwise, but Tarski's approach is not intended to solve the problem. It is to provide a replacement that does not have problems.[21] Second, the hierarchical solution appears to misdiagnose the problem. There are unproblematic cases of self-reference involving truth – such as this (true) one right now – so banning it outright is overkill. More generally, there is nothing particularly ungrammatical or categorically wrong with the liar sentence as such: it seems well formed; it has the right type of subject for its predicate, as opposed to something like "the Pope's chair is a prime number." If one wants to try to block the paradox by restricting the expressive power of the language, it should at least be by a more surgical incision.[22]

The most important problem for any hierarchy, though, is the question: how can a hierarchical theory be *true*, according to itself? True claims must be indexed to some level of the never-ending hierarchy, but the claim "all true claims must be indexed to some level of the hierarchy" cannot be so indexed. In a straightforward sense, then, a hierarchical

18 [Tarski, 1956a].
19 Increasingly sophisticated formal theories of truth have followed. For the state of the art, see [Halbach, 2014].
20 For a polemical account, see [Routley, 1979].
21 Cf. the *inconsistency theory* of truth [Azzouni, 2006; Scharp, 2013].
22 As Russell wrote of similar such solutions in set theory, they "seem to be created *ad hoc* and not to be such as even the cleverest logician would have thought of if he had not known of the contradictions" [Russell, 1959, p. 61].

theory of truth *is not true according to its own account*.[23] This is an extremely bad feature for a (putatively true) theory of truth to have. Cutting to the chase,

> The paradoxical aspect of Tarski's theory, indeed of any hierarchical theory, is that one has to stand outside the whole hierarchy even to formulate the statement that the hierarchy exists. . . . The paradoxes themselves are hardly less paradoxical than the solutions to which the logical community has been driven. *[Putnam, 1990, pp. 14, 17]*

The same problem arises with any solution to the paradoxes, via stratification or otherwise, that attempts to block quantification over all truths, all sets, all propositions, etc. The truth schema appears to talk about all propositions at once; but to say, à la Russell, that "No proposition may quantify over all propositions," is self-undermining. The prohibition prohibits itself [Priest, 2002a, ch. 9].

This problem – that the solution to a self-referential paradox is itself self-referentially undermining – is an instance of *revenge*, the observably repeated pattern in which the basic notion that a solution requires is not available by the lights of that very solution.[24] A neat sum of the history:

> It slowly dawned on us that it is unbelievably difficult to say anything at all about the aletheic paradoxes without contradicting oneself. *[Scharp, 2013, p. 2]*

Revenge is what makes the paradoxes paradoxes, strikingly different from other types of hard problems: when you try to solve the problem, *you get the same problem back!* You try to block self-reference, and find that you must self-refer to do so. You try to ban universal quantification, and find that you must universally quantify to do so. Some problems are impossible to solve simply due to their complexity. But the liar is not complex. It is the very simplicity of the problem that makes it so troubling; it is the endless resilience of the paradox that makes it so compelling.

Classical solutions would appear to maintain the law of excluded middle (LEM), p or $\neg p$. In this way, they maintain, with respect to Gödel's diagnosis, that for any property φ, the universe can be divided in to two categories, the φs and the $\neg\varphi$s. But this preservation of bivalence is largely an illusion, because the classical position is a *conditional*: *if* there were a universe, then it could be divided in two. But hierarchies can never be finished, never take full stock of themselves, on pain of a liar contradiction; since the classical position eschews all inconsistency, there cannot be a level from which to talk (truly) about all levels. On strictly classical grounds, there is no such thing as universal quantification of an absolutely general sort, because on pain of contradiction there is no completed hierarchy of truths or domain over which to quantify.[25] On pain of contradiction, the classical position reduces to this arresting claim: *There is no universe* [Halmos, 1974, p. 7].[26]

23 Fitch decried the "dim outlook" for philosophical metatheory since Tarski, because "there seems to be no final formal language adequate for dealing with its own semantical concept of truth" [Fitch, 1964, p. 397].

24 On revenge in general (but in the context of one proposal to deal with the paradoxes), see Priest [2006c, 2007].

25 See [Rayo and Uzquiano, 2006]. We return to this issue when we get to set theory Section 1.1.

26 At the close of his charming autobiography *I Want to Be a Mathematician*, Halmos credits to himself the abbreviation "iff" and the little square "tombstone" marker at the end of proofs.