1 Introduction

This Element aims at an in-depth, philosophically oriented, readable yet formally careful study of the semantic concept of *logical consequence* (henceforth, LC) – arguably, the most central concept of logic and one of the most vibrant topics of discussion in contemporary philosophy of logic. The Element seeks to get to the bottom of substantial philosophical questions concerning LC, questions that are often overlooked or taken to be unsolvable. As a result, it offers a unique perspective on LC.

1.1 The Idea of Logical Consequence

The idea of LC, or logical inference, is the idea of what follows logically from what. Given a collection of sentences $\Gamma = \{S_1, S_2, \ldots, S_n, \ldots\}$, $n \geq 0$, and a sentence, S, either S follows logically from Γ – S is a LC of Γ – or not. For example, the sentence “The number 1 has a successor” is a LC of the sentence “Every number has a successor,” and the sentence “Biden won the election” is a LC of the sentences “Either Trump won the election or Biden won the election” and “Trump did not win the election.” But “Every number has a successor” is not a LC of “The number 1 has a successor,” and “Biden won the election” is not a LC of “Either Trump won the election or Biden won the election.”

To understand the idea of LC in depth is to understand, among other things:

- In virtue of what a given sentence is a LC of another sentence (other sentences)
- What is the role of LC in knowledge
- Why and how LC works in the world (why making logical errors in, say, the design of an airplane can cause it to crash)
- What are the distinctive characteristics of LC
- What laws and principles govern or underlie LC
- Whether there is a precise method for determining that S is (or is not) a LC of Γ
- Whether there is just one kind of LC or many
- How the idea of LC relates to that of consequence in general
- What the structure of a system of LC is or could be
- What are the mathematical properties of LC (system[s] of LC)
- What is the source of normativity of LC

Logical consequence is sometimes thought to be a trivial relation, one that is obvious and cannot provide genuinely new knowledge. This view, however, is likely to be mistaken. Many mathematical theorems, which are LCs of mathematical axioms, provide genuine and indeed (at least originally) surprising new knowledge, as surprising as much of the new knowledge acquired by science.

Indeed, in science, too, logical inferences are used to arrive at far from obvious predictions and discoveries. Although here, unlike in mathematics, new predictions and discoveries are normally arrived at through a joint use of logical and nonlogical inferences, it is highly unlikely that logical inferences employed to arrive at new results do not play a substantial role.

One might object on the ground that the primitive logical rules used in such inferences are obvious or at least very simple. There is, however, nothing in the idea of logical rules that requires primitive rules to be obvious or simple: The choice of which rules to treat as primitive is largely pragmatic. More importantly, even when the primitive rules are obvious (or simple), combinations of such rules are often far from obvious.

1.2 Proof-theoretic and Semantic Approaches to Logical Consequence

The concept of LC is commonly understood both in terms of *proof* (S is a LC of Γ iff [if and only if] there is a proof of S from the sentences of Γ) and in terms of *truth* (S is a LC of Γ iff the truth of all the sentences of Γ guarantees the truth of S). Accordingly, the theory of LC has two branches: proof theory, which systematizes the notion of LC in terms of proof; and semantics or model theory, which systematizes it in terms of truth. The common proof-theoretic and semantic symbols for LC are "⊢" and "⊨," respectively. If Γ is a collection of sentences (premises) and S is a sentence (conclusion), "Γ⊢S" says that S is *logically provable* or *derivable* from the sentences of Γ and "Γ⊨S" says that the truth of the sentences of Γ, assuming that they are all true, *logically guarantees* the truth of S. In both cases, we also say that the consequence (inference) is *logically valid*.

Historically, systems of LC began as proof systems, centering on logical rules of proof such as modus ponens ($S_1 \supset S_2$, $S_1 \vdash S_2$) and universal instantiation ($(\forall x)\Phi x \vdash \Phi a$), where Φ is a formula and *a* is an individual constant (name of an individual). Semantics was often used in informal explanations but was not part of the logical system. Starting in the early twentieth century, however, logicians integrated the semantic account into their logical systems. Today, logical systems are usually divided into three fully developed subsystems: syntax, proof theory, and semantics (or model theory). Syntax offers a precise formulation of the language of a given logical system, proof theory offers a precise formulation of logical *provability* (or *derivability*) for that system, and semantics or model theory offers a precise formulation of LC for the given system in terms of models and truth. The term "LC" is commonly used to indicate the semantic or model-theoretic version of "following logically." In this Element we study the semantic theory of LC.

1.3 Historical Origins of the Semantic Approach to Logical Consequence

From the outset, logicians used semantic notions to explain and motivate their development of modern logic, though at first largely informally. For example, Frege, whose *Begriffsschrift* (1967 [1879]) is often used to mark the birth of modern logic, said that all of the axioms of his logical systems are *true* and its rules *truth-preserving*. The largely informal use of semantic notions continued until the publication of Tarski's semantic definition of LC (1983 [1936a]). Even Gödel's celebrated proofs of the completeness of standard first-order logic (1986 [1929]) and the incompleteness of stronger logical systems (1986 [1931], applied to logic) are of this kind. Whereas today, Gödel's completeness theorem is understood to show the extensional equivalence of the proof-theoretic and semantic notions of LC in standard first-order logic, originally it was presented as establishing the completeness of the proof-theoretic method of this logic in light of our informal understanding of logical inference/consequence in terms of truth. (By "*standard*" first-order logic, I mean, in this Element, "having the standard logical constants (ℓcs): '~,' '&,' '⊃,' '≡,' '=,' '∃,' '∀,' and ℓcs defined from these.")

Some metalogical results prior to Gödel were already formulated explicitly in model-theoretic terms (e.g., Löwenheim's 1915 [1967 (1915)] and Skolem's 1920 [1967 (1920)] theorems). And Hilbert made important contributions to the development of model theory at the turn of the twentieth century (see Hilbert 1950 [1899], Hilbert and Ackerman 1950 [1928]). Still, model theory was not an integral part of modern logic at the time. It was only after the development of the semantic definition of LC by Tarski (1983 [1936a]) that logical semantics or model theory became one of the cornerstones of modern logic, alongside proof theory.

1.4 Consequence in General and Logical Consequence

There are many kinds of consequence. For example, "*a* is a physical body; therefore, *a* does not move faster than light" is a *nomic* physical consequence. Logical consequence is a consequence relation/concept of a special kind.

The general relation of consequence – S is a consequence of Γ – is a relation of transmission or preservation of truth from Γ to S. The weakest type of consequence is:

Material Consequence (MC)

S is a MC of Γ iff: if all the sentences of Γ are true, S is true,

where truth is truth *simpliciter* – that is, *material* truth in the sense of truth-in-the-actual-world.

In symbols: *S is a MC of Γ iff* $T(\Gamma)\supset T(S)$.[1]

[1] "⊃" is the truth-functional conditional.

Material consequence is an extremely weak consequence. Generally, all that is required for S to be a MC of Γ is that either one of the sentences of Γ is materially false or S is materially true. Both "Tarski is a US president; therefore, Biden is a logician" and "Tarski is a logician; therefore, Biden is not a logician" are MCs. Not all consequences, however, are so weak. Nomic consequences are stronger than MCs, but there are still stronger types of consequence. Logical consequence is a stronger type of consequence. This is reflected in the traits commonly attributed to it: strong generality, necessity, formality, topic neutrality, certainty, normativity, and so on. (Nomic consequences are not formal or topic neutral and their generality, necessity, certainty, and normativity are weaker than those of LCs.) Different types of consequence are due to different types of relations between their premises and conclusion. While nomic physical consequences are due to physical relations between physical constituents of their premises and conclusion, LCs are due to logical relations between logical constituents of their premises and conclusions.

The notion of LC has been expanded to logics of multiple kinds, such as modal and relevance logics. Here we focus on the relation of LC in what is often called *predicate*, *formal*, or *mathematical logic*, which is the main modern successor of Aristotelian logic and is widely considered a core logic.[2] Even within the boundaries of this type of logic, however, the scope of LC is, philosophically, an open question.

1.5 Philosophical, Mathematical, and Linguistic Interest in Logical Consequence

The theory of LC is significant for several disciplines, principally philosophy, mathematics, and linguistics. In this Element I focus primarily on the philosophical character of LC as it is studied in mathematics and philosophy. For important work on issues related to LC within linguistics, see, for example, Montague (1974), May (1985), and the overview in Peters and Westerståhl (2006).

2 The Semantic Definition of Logical Consequence and Its Roots in Tarski

The semantic definition of LC has its roots in Tarski's paper, "On the Concept of Logical Consequence" (1983 [1936a]). Reading the first work in which a new theory/definition is introduced often provides important insight into the motives, goals, and considerations that led to its development. This is true of

[2] By calling this logic "mathematical logic," I do not mean that it is applicable only, or even primarily, to mathematics.

Tarski (1983 [1936a]). Tarski, however, disliked extended philosophical discussions, preferring to pursue mainly the formal aspects of his ideas. As a result, it is left for us to work out the philosophical content of some of Tarski's ideas and evaluate their significance.

2.1 Tarski's Route from Truth to Logical Consequence

Tarski's semantic definition of LC (Tarski 1983 [1936a]) followed upon, and extensively employed, his semantic definition of truth (1983 [1933]). This definition set out to provide a "*materially adequate and formally correct definition of the term 'true sentence'*" (Tarski 1983 [1933]: 152). By a "materially adequate" definition, Tarski meant a definition that captures the intended content of the defined concept, which, in the case of truth, he identified with the "so-called *classical* conception of truth ('true – corresponding with reality')" (Tarski 1983 [1933]: 153). By a "formally correct" definition, he meant a definition that avoids paradoxes (such as the Liar paradox) and satisfies general formal requirements on definitions.

One may wonder how a definition of truth can serve as a basis for a definition of a distinctly logical (or rather metalogical) concept such as LC, and what motivated Tarski, a mathematical logician, to define a distinctly philosophical concept such as truth in the first place. One conjecture, due to Vaught, is that Tarski's interest in truth had to do with the state of logic, and in particular metalogic, in the early decades of the twentieth century: "Tarski had become dissatisfied with the notion of truth as it was being used [in meta-logic at that time]" (Vaught 1974: 161). The notion of truth had been widely used in metalogic informally. Indeed,

> it had been possible to go even as far as the completeness theorem by treating truth (consciously or unconsciously) essentially as an undefined notion – one with many obvious properties . . . But no one had made an analysis of truth, not even of exactly what is involved in treating it in the way just mentioned . . . [T]his whole state of affairs . . . cause[d] a lack of sure-footedness in metalogic. (Vaught 1974: 161)

Vaught's point throws light on two familiar, yet quite unusual, features of Tarski's definition of truth: (i) Tarski defined truth only for *logical* languages – "formalized languages of the deductive sciences" (Tarski 1983 [1933]: 152); (ii) Tarski's definition of truth is focused on the *logical* structure (or logical content) of sentences: For each primitive ℓc – that is, a constant that determines a primitive logical structure – but not for structure-generating constants of other types (e.g., causal constants such as "because"), there is a special entry in Tarski's definition of truth that specifies the truth conditions of sentences with that logical structure. The recursive character of the definition enables it to define truth for all logically structured sentences in a finite number of steps. Given these distinctive characteristics of Tarski's definition of truth, it is not surprising that it provides a natural basis for definitions of logical, or metalogical, concepts such as LC.

Tarski classified "truth," along with "denotation" ("reference"), "satisfaction," and "LC," as *semantic* concepts. What are the distinctive characteristics of semantic concepts for Tarski? Some semantic concepts are defined in terms of other semantic concepts, for example, "truth" in terms of "denotation" ("reference") and "satisfaction." But being semantic, for Tarski, is not simply a matter of definability. His classification of concepts as semantic has to do with their *content*, and in particular with a specific aspect of their content:

> A characteristic feature of the semantical concepts is that they give expression to certain *relations* between the *expressions of language* and the *objects* about which these expressions speak. (Tarski 1983 [1933]: 252, my emphasis)
>
> We . . . understand by semantics the totality of considerations concerning those concepts which, roughly speaking, express certain connexions between the expressions of a language and the objects and states of affairs referred to by these expressions. (Tarski 1983 [1936b]: 401)

Semantic concepts, thus, are not concepts that deal with just any aspect of meaning. They are concepts that deal, either directly or indirectly, with a particular aspect of meaning, namely, the *relation between language and the world*.[3]

For many concepts commonly viewed as semantic, this characterization is natural. Some of these express relations between language (linguistic expressions) and world (objects) directly. "Reference" ("denotation") and "satisfaction" fall under this category. The individual constant (proper name) "Biden" refers to the person (object in the world) Biden. The person (object in the world) Biden satisfies the 1-place predicate "x-is-president-of-the-US-in-2021," the pair of numbers $<1,2>$ satisfies the 2-place predicate "$x<y$," and so on. Other semantic concepts satisfy Tarski's characterization indirectly. *Truth* is such a concept. Tarski, as we have already seen, characterized his concept of truth as a correspondence concept. Truth is a property of sentences, but a sentence has this property only if a certain relation between it and the world holds. This relation, according to Tarski, is the *content* of the concept of truth, and a definition of truth is *materially adequate* only if it captures this content.[4]

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3 Tarski did not specify the scope of "world" ("object," "state of affairs"). I think it is reasonable to presume that he viewed its scope as fairly broad (e.g., as including mathematical objects), yet as having no definite boundaries. "World" seems to have been, for him, an intuitive, common sensical, pretheoretic notion, leaving room for a variety of precisifications.

4 (i) Tarski also said that he reduced all semantic concepts to "structural-descriptive" concepts – concepts that belong to the "morphology" of language (Tarski 1983 [1933]: 252) and refer to linguistic expressions by describing their structure. But Tarski saw no conflict between the structural-

When it comes to the semantic concept of LC, however, Tarski's understanding of "semantic concept" is philosophically intriguing. Philosophers often think of logic as having to do with language and concepts alone, not with the world. But if LC, like truth, is a semantic concept in Tarski's sense, then it, too, has to do with the relation between language and the world. Yet how is LC related to the world? Logical consequence is a relation between linguistic entities (sentences). What in the world, or what aspect of the world, does this relation correspond to? Could we say that LC is a semantic concept because, and only because, it is defined in terms of semantic concepts, that is, concepts that have to do with the relation between language and the world, but it itself has nothing to do with this relation, and in particular with the world? Well, if the *main* concepts in terms of which LC is defined are of a kind whose *distinctive* characteristic is that they relate language to the world, then LC itself is likely to have this characteristic, and we are back to the idea that LC is a correspondence concept (of some kind). This puzzling issue requires an in-depth investigation, and we will return to it in Section 4, after we have obtained a deeper understanding of LC and related issues.

Another distinctive feature of semantic concepts, as conceived by Tarski, is that they are *metalinguistic*. To say that a concept is metalinguistic is to say that it refers to, or concerns, linguistic entities. Since semantic concepts concern linguistic entities (albeit in their relation to the world), they are metalinguistic. Given a language L, the concepts of truth and LC for L belong to its metalanguage, ML, the concepts of truth and LC for ML belong to MML, and so on.[5]

descriptive, that is, linguistic, definition of semantic notions and their correspondence content. Referring to the Medieval principle of *suppositio materialis* Tarski pointed out that we can express the relation between words and objects (world) linguistically, as a relation between words that refer to objects and names of (structural-descriptive expressions that refer to) these words.

(ii) Note the difference between this use of "material" and its use in Section 1.4. "Material adequacy" means "capturing the intended content of a given concept"; "material (truth/consequence)" means "truth/consequence that holds in the actual world." In this Element I use "material" in both senses. It will be clear from the context which sense is intended.

[5] Tarski's treatment of the semantic concept of truth as a metalinguistic concept has been widely criticized on the grounds that (i) it is ad hoc, (ii) it relativizes truth to language, and (iii) it diverges from natural language that has only one truth predicate. Without getting into these controversies and without referring to what Tarski himself said, let me briefly suggest responses to these criticisms that are relevant to a contemporary understanding of truth and LC. (i) It is inherent in the correspondence conception of truth that to determine whether a given sentence S is true, we need to transcend this sentence to a standpoint from which we can see (a) the sentence S, (b) its target in the world, and (c) the relation between them. This is exactly the standpoint of a Tarskian metalanguage. (ii) Tarski's definitions of truth and LC are technically relative to language, but it is significant that the entries for the ℓcs in the definition of truth are the same for all languages, and the definition of LC is also essentially the same for all languages. (iii) The natural-linguistic perspective is just one philosophical perspective on truth and LC, and not necessarily the most important one.

2.2 The Need for a Non-proof-theoretic Definition of Logical Consequence

Tarski emphasized the need for a definition of LC as it is used in everyday life, mathematics, and empirical science.[6] But by 1936 this concept already had a definition, namely, a proof-theoretic definition, one formulation of which is:

Proof-theoretic Definition of Logical Consequence

Given a logical system $\mathscr{L}$, a language L of $\mathscr{L}$,[7] a sentence S of L, and a collection Γ of sentences of L:

S is a LC of Γ iff there is a proof of S from Γ,

where a proof of S from Γ is a finite sequence of sentences, $<S_1, \ldots, S_n = S>$, such that for every $1 \leq i \leq n$, either (i) S_i is a member of Γ, or (ii) S_i is an axiom of $\mathscr{L}$, or (iii) S_i is provable from $S_1, \ldots, S_{i-1}$ by a rule of proof of $\mathscr{L}$.

The need for a new definition of LC arose from the limitations of the proof-theoretic definition, revealed by Gödel's 1931 incompleteness theorem (1986 [1931]; see Section 4.7). This theorem shows that the proof-theoretic concept of LC is significantly narrower than the concept informally used by mathematicians and others.[8] From this, it follows that to adequately capture the full concept of LC, we need a new definition.

2.3 Fundamental Adequacy Conditions: Truth Preservation, Necessity, Formality

Tarski's paper, as I emphasized earlier, provides important philosophical insights into the semantic concept/relation of LC. These include two fundamental philosophical features of LC: *necessity* and *formality.* Tarski used these features as guidelines in his search for an adequate definition of this concept: An adequate definition must render the relation of LC both necessary and formal. The claim that LC is fundamentally necessary and formal requires a critical explanation and examination, which I will give later on. At this point, we would like to understand what Tarski himself said about these guidelines and how they constrained his definition of LC. Using "Γ⊨S" for "S

[6] Although he was aware that in systematizing an intuitive, informally used concept, one cannot be completely faithful to all of its uses.

[7] In this Element, a logical system $\mathscr{L}$ has a fixed collection of ℓcs, axioms (axioms schemas), and rules of proof. A language L adds non-ℓcs to a given $\mathscr{L}$. All Ls (for $\mathscr{L}$) share the same ℓcs, axioms, and rules of proofs (–$\mathscr{L}$'s) but differ in their non-ℓcs.

[8] While the standard first-order proof-theoretic concept of LC does coincide with the intended concept when the latter is limited to standard first-order languages, the full intended concept of LC, Tarski assumed, is not limited to such languages, and as such is essentially broader. (This explanation is simpler and more straightforward than in Tarski 1983 [1936a], but the two come to the same thing.)

is a LC of Γ," we can formulate these guidelines for (or conditions on) an adequate definition of LC as follows:

> *Necessity*: An adequate definition of LC renders LCs *necessary*, that is, if Γ⊨S, then *necessarily*, if all of the sentences in Γ are true, S is true. In symbols: Γ⊨S ⊃ **Nec**[T(Γ)⊃T(S)].
>
> *Formality*: An adequate definition of LC renders LCs *formal*, that is, if Γ⊨S, then *formally*, if all the sentences in Γ are true, S is true. In symbols: Γ⊨S ⊃ **For**[T(Γ)⊃T(S)].

A third condition, underlying these two, is:

> *Transmission/Preservation of Truth:* An adequate definition of LC transmits/preserves *truth* (simpliciter) from Γ to S, that is, if Γ⊨S and all members of Γ are *true*, S is *true* too. In symbols: Γ⊨S ⊃ [T(Γ)⊃T(S)].

We will further discuss the third condition in later sections. Here we follow Tarski in focusing on necessity and formality.

Tarski did not provide any explanation of necessity, treating the necessity requirement as a straightforward pretheoretical requirement. He did provide a partial explanation of formality, though it is not altogether clear what the main point was. He began by saying that (i) LC is formal in the sense of being "uniquely determined by the *form* of the sentences between which it holds" (Tarski 1983 [1936a]: 414). This may lead us to think that formality, for him, was a *syntactic* feature of LC. But what Tarski said next shows that he had in mind something that goes beyond syntax: (ii) LC "cannot be influenced in any way by empirical knowledge, and in particular by knowledge of the objects to which the [sentences in Γ and S] refer" (*ibid.:* 414–415); (iii) LC "cannot be affected by replacing the designations of the objects referred to in these sentences by the designations of any other objects" (*ibid.:* 415).

Based on (iii), we may say that a definition of LC satisfies the formality requirement only if it renders LC *invariant* under uniform replacements of non-ℓcs by constants (of the same syntactic type) that denote different objects. If we use "*" to indicate such a uniform replacement, (iii) says that the formality condition on an adequate definition of Γ⊨S is satisfied iff for every replacement * of the non-ℓcs (of the given language), T(Γ*) ⊃ T(S*). This rendition of the formality condition naturally suggests a *substitutional* definition of LC (where "substitution" stands for "uniform replacement of non-ℓcs"), and Tarski's next step was, indeed, to formulate, and then reject, such a definition.

2.4 Inadequacy of the Substitutional Definition

We may formulate Tarski's substitutional definition as follows:

Substitutional Definition of Logical Consequence

Let $\mathcal{L}$, L, Γ, and S be as in the proof-theoretic definition of LC. Using "$\vDash_{SB}$" for the substitutional relation of LC, we define:

> *$\Gamma \vDash_{SB} S$ iff for any uniform substitution, *, of the non-ℓcs in Γ and S by non-ℓcs of L of the same syntactic types: T(Γ*) ⊃ T(S*), that is, if all the sentences of Γ are true under *, S is true under **

where "true" means "true simpliciter," that is, "materially true" ("true in the actual world").

Examining this definition, Tarski concluded that while it sets a necessary condition for LCs, it does not set a sufficient condition. The substitutional definition is satisfiable not just by LCs, but also by non-LCs. The reason is that whether the substitutional test works depends on the richness of the language L. If L does not have enough non-ℓcs to generate counterexamples for all non-LCs, then some non-LCs of L will be pronounced LCs by this test.

Example: Let the entire nonlogical vocabulary of L consist of "Logician," "Tarski," and "Frege." Consider the consequence:

(1) Tarski is a logician; therefore, Frege is a logician

Clearly, (1) is merely an MC, but it passes the substitutional test. For any uniform substitution of "Logician," "Tarski," and "Frege" by non-ℓcs of L (of the same syntactic type),

$$\mathrm{T}[\mathrm{Logician}(\mathrm{Tarski})]^{*} \supset \mathrm{T}[\mathrm{Logician}(\mathrm{Frege})]^{*}.$$ [9]

So, the substitutional test fails. The nonlogical vocabulary of L is too impoverished to provide counterexamples for all nongenuinely-LCs. Tarski concluded that the substitutional definition is inadequate:

> The [substitutional definition] could be regarded as sufficient for the sentence [S] to follow [logically] from the class [Γ] only if the *designations [names] of all possible objects* occurred in the language in question. This assumption, however, is fictitious and can never be realized. (Tarski 1983 [1936a]: 416, my emphasis)

We cannot assume that given an arbitrary language L, its nonlogical vocabulary is sufficiently rich to provide an adequate substitutional test of LC.

Another problem with the substitutional definition of LC, not mentioned by Tarski, is that generally, it takes into account only the truth of sentences in the *actual* world. Consider the consequence:

(2) There is exactly one individual; therefore, there are at least two individuals.

[9] Here, [Logician(Tarski/Frege)]*=Logician*(Tarski*/Frege*).