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An Indefinite Excursion in Operator Theory

Geometric and Spectral Treks in Kreĭn Spaces

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To Tiberiu Constantinescu (1955–2005) and Peter Jonas (1941–2007)

who shared with me the joy of trekking through

the indefinite realm of operator theory

as well as

To my beautiful and generous country, Roumania

Contents

Preface

The idea of writing a monograph on operator theory on indefinite inner product spaces occurred to me about twenty years ago, after teaching a graduate course on the spectral theory of definitisable operators at the Faculty of Mathematics and Informatics of the University of Bucharest. With a lot of enthusiasm I wrote the core of the chapters on spectral theory and invited Heinz Langer to join me in this enterprise. At the beginning, Heinz showed his interest and encouraged me by performing a careful reading of those notes and providing very pertinent observations, but in the end he declined my invitation. That left me alone in this enterprise and offered me more freedom in choosing the subjects to be included, but this path proved to be dangerous in the end, as I explain below.

After performing research on the geometry of Kreĭn spaces and the spectral theory of selfadjoint definitisable operators in the 1980s as part of my PhD programme, I became more focused on studying dilation theory of operators on Kreĭn spaces and started to work together with my friend and collaborator Tiberiu Constantinescu. Dilation theory is a heterogeneous domain full of concepts and results that may look connected in some way but not explicitly, and because of that it became clear, at least to us, that it needs some unification. Considering dilation theory from a rather general perspective, we soon realised that one way to unify most of dilation theory is to consider Hermitian kernels and their linearisations, or Kolmogorov decompositions, which turn out to be yet another face of reproducing kernel Kreĭn spaces. Moreover, using some older ideas from mathematical physics, we also understood that some invariance under group, or $*$-semigroup, actions would open rather large avenues and connections with problems of a wider interest.

So, I made a very general and ambitious plan for a book that should contain and explain the extraordinary connections that I have observed between the geometry of Kreĭn spaces, the spectral theory of their linear operators, the dilation theory of Hermitian kernels and reproducing kernel Kreĭn spaces. The material grew over the years, but challenging problems showed up because important parts of this programme had not yet been investigated. Then, at some moment I had to admit that this plan was too ambitious and I should split it: on the one hand, I should dedicate my energy and time to exploring the new territory at the level of research articles and, on the other hand, I should reduce the plan of the book to a more realistic one. Therefore, I eliminated most of the dilation theory, harmonic analysis on Kreĭn spaces, and Hermitian kernels from the plan and I ended up with what is now this monograph: an excursion in the realm of operator theory on Kreĭn spaces from the point of view of the interplay of geometry and of spectral theory.

* * *

This monograph is a gentle and modern introduction to spaces with indefinite inner products and their operator theory and it is supposed to continue and complement the two existing monographs on this subject, that of J. Bognár (1974) and that of T. Ya. Azizov and I. S. Iokvidov (1986), and also gather some important results that were not included in those books or were obtained after 1980. Operator theory on indefinite inner product spaces has developed rapidly during the last thirty years and, unfortunately, there are many directions of research that have been explored and many interesting results that have been obtained that I had to leave out. This monograph is a selection of those topics that I consider to be representative for this theory and, of course, this selection is subjective.

Operator theory on Kreĭn spaces is, of course, a part of operator theory and a natural question is what novelty can it bring into a domain that is already so diverse and sophisticated. A possible answer to this question comes from the geometry of Kreĭn spaces, when compared to that of Hilbert spaces, and here lie both the power and the weakness of this theory. The simplest, intuitive, view of a Kreĭn space is as an infinite dimensional and complex analogue of Minkowski space, the space-time that lies at the foundation of relativistic physics. In a classical Minkowski space one encounters both positive and negative "lengths" of vectors, and also null "lengths" of nontrivial vectors. In the case of Minkowski space, the latter vectors, called neutral, make the so-called light cone which separates space-time into two distinct regions, with rather different physical interpretations. So, in a Kreĭn space, one has all these "anomalies" and even more, due to the intricate geometry of indefiniteness combined with the topological complications that show up in infinite dimensional spaces.

Continuing with this analogy, let us recall that the class of displacements that leave invariant Minkowski space is made by the Lorentz group $O(3,1)$, which is not compact. Reasoning by similarity, we expect that the group of unitary operators in a Kreĭn space will play a central rôle. This indeed happens but, even more than that, in a genuine Kreĭn space the natural generalisation of a unitary operator leads to unbounded operators. This simple fact, that appears right from the beginning, of having to deal with unbounded isometric operators when considering indefinite inner product spaces, gives us just a pale idea of the novelty and the difficulties of operator theory in Kreĭn spaces.

From the point of view of applications, spectral theoretical aspects prevail in operator theory but, from my experience, the best way to explain the difficulties and, why not, the beauty, of the spectral theory of operators in Kreĭn spaces is to connect it with the more intricate geometry that comes together with indefiniteness. What the theory of linear operators on indefinite inner product spaces brings into play is a certain inner symmetry that, when emphasised, may or may not solve many difficulties, but at least it brings more geometry to the problems that we deal with.

The idea of merging the geometry of indefinite inner product spaces with the spectral theory of linear operators on these spaces is substantiated in Chapter 9 which presents a panoply of situations under which operators, or families of operators, have invariant maximal semidefinite subspaces. This problem is central to the theory of operators in indefinite inner product spaces and turns out to catch the core of this theory. Although it is not my aim to extensively explore applications of operator theory on indefinite inner product spaces to other domains of mathematics, we dedicate Chapter 10 to applications of invariant maximal semidefinite subspaces to interpolation for meromorphic functions, to Nehari type problems, and to Hankel type operators. In this way, the importance and the technical difficulties encountered when dealing with problems on invariant maximal semidefinite spaces provide a consistent justification for further exploration of the spectral theory of selfadjoint operators and unitary operators, on the one hand and, on the other hand, exploration of contractions and their generalisations, quasi-contractions, from both spectral and geometric points of view. It comes then as no surprise that, in order to obtain satisfactory results on the spectral theory of selfadjoint or unitary operators on Kreĭn spaces, imposing more technical assumptions is unavoidable. In the final chapters we employ the condition of definitisability in the sense of Heinz Langer which has proved to be remarkably successful. At first glance, the concept of definitisability may look too narrow and hence we felt obliged to justify its power by showing how useful it is in producing rather strong results on quasi-contractions. In this way we go back, from spectral theory to geometry, thus closing the circle.

There is one more characteristic of the approach that was used in this book which, at first glance, might not be visible. The applications that are included in this volume refer mostly to problems related to complex functions: the realisation theorems from Chapter 6 which concern kernels of holomorphic functions of Nevanlinna, Carathéodory, and Schur type as well as the problems of interpolation of meromorphic functions that occupy an important part of Chapter 10. But this is only one face of this characteristic. The other face can be seen when taking a closer look at the techniques that allow us to perform spectral theory and that require sophisticated results of holomorphic functions, like the Herglotz representations theorem, for example. So, in this monograph, the theory of complex functions plays a central rôle not only in applications but also in the essential tools that allow us to approach spectral theory in its large diversity.

In dealing with Kreĭn spaces there are, historically, two different points of view. One point of view, that was used, for example, by J. Bognár [18] and that we follow in this book, is to consider an inner product space $(\mathcal{X}; [\cdot,\cdot])$ onto which a certain Hilbert space topology that makes the indefinite inner product jointly continuous can be defined, as in Theorem 1.4.1. In this respect, the associated concepts such as fundamental symmetry, fundamental decomposition, and fundamental norm, are not fixed and they can be changed, within well specified classes, according to the requirements of the problems we are interested in. I consider that this approach offers a lot of flexibility and points out the prevailing properties of the indefinite inner product and the underlying geometry. It also points out the fact that the topology on a Kreĭn space is an extrinsic object and not intrinsic, as is the case with Hilbert spaces.

The other point of view, which was used by T. Ya. Azizov and I. S. Iokhvidov [10], is to start with a Hilbert space $(\mathcal{H}; \langle\cdot,\cdot\rangle)$ onto which a symmetry J, that is, a bounded linear operator on $\mathcal{H}$ such that $J = J^* = J^{-1}$, equivalently, J is a unitary selfadjoint operator on $\mathcal{H}$, is given and then introduce the indefinite inner product $[x, y] = \langle Jx, y\rangle$. This approach is used in many articles and has some merits but it can also lead to confusion and to some difficulties in understanding the geometry of a Kreĭn space. Moreover, as the results in Chapter 6 show, for most applications we do not have an underlying Hilbert structure available beforehand and we have to build one from the indefinite structure by an inducing construction. In this respect, we have to draw attention to the fact that the notation and the terminology of these two approaches are rather different.

In this monograph we intend to offer a presentation that, on the one hand, can be read and understood by a wider audience and, nevertheless, has some unity and harmony and, on the other hand, includes some of the most comprehensive results and the most relevant examples that point out the power and the applicability of operator theory on indefinite inner product spaces. Thus, we gradually explore the geometry of indefinite inner product spaces and their linear operators and introduce the spectral theoretical aspects only after sufficient experience and examples have been given.

We tried to increase the level of completeness of this book by avoiding sending the reader in search of books or articles that may be difficult to find. On the one hand, by imagining this monograph as an excursion, on a few occasions we take the freedom to step aside and we dedicate some pages to a careful presentation of the technical results that are needed, for example, the Herglotz theorems on representations of holomorphic functions mapping the upper half plane into itself, the Stieltjes inversion formulae, and the fixed point theorems in locally convex spaces. On the other hand, we want to make this book useful not only for the research mathematician, but also for the interested physicist or engineer, and especially for graduate students. For this reason, we have added nine appendices that review the basics of general topology, measure theory, topological vector spaces, functional analysis, complex functions, operator theory in Banach spaces with an emphasis on operator theory in Hilbert spaces, for both bounded and unbounded operators, and Fredholm theory. Of course, there are other prerequisites that the reader is supposed to be aware of, for example, basic linear algebra, a good command of differential and integral calculus at the level of advanced calculus, and some basic abstract algebra. Anyhow, before embarking on this excursion, readers are kindly advised to take a quick look at the appendices and check that all these concepts and results are present in their equipment.

On one occasion in Chapter 2, after performing a rather elaborate exploration of lifting of contractions in connection with R. S. Phillips's theorem of extensions of pairs of semidefinite subspaces to maximal ones, the excursion takes a side path in order to apply these results to extensions of dual pairs of accretive operators to maximal dual pairs and to extensions of positive operators to positive selfadjoint operators in Hilbert space. In this way we make available an older manuscript [28] that was not published. Also, there is a deeper connection here: the theory of positive selfadjoint extensions of positive operators was initiated by Mark G. Kreĭn in [92] and the original proof of Phillips's theorem [120] was inspired exactly by that article.

Operator theory on indefinite inner product spaces was originally motivated by and obtained remarkable success when applied to the spectral theory of ordinary and partial differential equations with certain symmetry properties. In this excursion we do not visit this kind of application because it would require a rather long preparation that would deviate considerably from the main course and would make this book too large. Although we think that a monograph dedicated to the spectral theory of ordinary and partial differential equations that employs techniques of operator theory on indefinite inner product spaces is necessary, this should be the topic of another book and, most likely, will be the project of another author.

* * *

I have to express my gratitude to many mathematicians who over the years influenced my research in operator theory on indefinite inner product spaces which eventually led me to write this monograph. As a student, I learned the basics of operator theory from Ion Colojoară, the basics of functional analysis from Ciprian Foiaş, then more sophisticated operator theory from Constantin Apostol and Dan-Virgil Voiculescu, and the basics of operator algebras from Şerban Strătilă. As a junior researcher in Bucharest, I was introduced to operator theory on indefinite inner product spaces by Grigore Arsene who at that time was in the group of mathematicians who worked in dilation theory. Then, I met Heinz Langer and Peter Jonas, from whom I learned a lot about the spectral theory of operators in spaces with indefinite inner product. Although my proposed collaboration with Heinz Langer on this monograph did not happen in the end, he made a careful reading of the chapters on spectral theory and provided valuable observations. An important part of the research in dilation theory in indefinite inner product spaces, which is present only to a very small extent in this monograph, was done in collaboration with my friend Tiberiu Constantinescu. Finally, by working in the domain of operator theory on indefinite inner product spaces I met more and more mathematicians and I became a part of the larger family of operator theorists that from time to time gathered in professional meetings, especially during the series of conferences on operator theory held in Timişoara, Roumania.

During the last year, I distributed the draft of this monograph to some of the specialists in operator theory on indefinite inner product spaces and I got very important feedback from them. Jim Rovnyak read very carefully the first six chapters and made valuable corrections and observations. Aad Dijksma provided a long list of corrections and remarks covering almost the whole book while Henk de Snoo drew my attention to some results that are not contained in this book but that deserve to be mentioned in the notes. Michael Kältenback provided another long list of corrections and observations on all chapters which helped me improve the presentation considerably. My former student Serdar Ay provided a list of corrections of the appendix. I want to express my deep gratitude to all of them.

I want to thank my parents, Elena and Ştefan, who took good care of me in difficult times and supported my decisions throughout their lives, although I was never able to explain to them what I am really doing. Finally, I want to thank my family, my wife Cristina and my children Alexandra and Sabin, who always supported me and, although we encountered difficult situations, they accepted that I needed to spend time and effort that kept me away from them, while I was wandering through the indefinite realm of operator theory.

Aurelian Gheondea — Bucharest, August 2021