Chapter 1

Inner Product Spaces

The first section of this introductory chapter contains more definitions and terminology than results. Although many of the definitions used in connection with inner product spaces are encountered in Hilbert space theory, there are also many new definitions that are specific to indefinite inner products. Many of these concepts come from the more general setting of duality theory but, in the context we use, they have special significance.

We then introduce the weak topology and discuss the existence of topologies making the inner product separately or jointly continuous. Among the inner product spaces admitting a normed topology that makes the inner product jointly continuous, an important rôle is played by Kreĭn spaces. In this chapter we consider only a general characterisation and leave to the next two chapters a more detailed description of their geometry. More tractable indefinite inner product spaces are the Pontryagin spaces, which we describe from both geometric and topological points of view.

One of the major differences between Kreĭn spaces and Hilbert spaces comes from the strong topology. In a Hilbert space, the strong topology is intrinsic to the inner product while, in the case of a genuine Kreĭn space, that is, one that is not of Pontryagin type, the strong topology has to be introduced independently. As a middle of the road we find the Pontryagin spaces which, although sharing similarities with Kreĭn spaces, have an intrinsic strong topology and hence these spaces are closer to Hilbert spaces.

Given an indefinite inner product space, a problem that often appears in applications is whether it can be "embedded" into a Kreĭn space. We give some characterisations of the affirmative case and some examples of the negative case.

1.1 Basic Definitions and Properties

Let $\mathcal{X}$ be a complex vector space. An *inner product* on $\mathcal{X}$ is, by definition, a mapping $[\cdot,\cdot]\colon \mathcal{X}\times\mathcal{X}\to\mathbb{C}$, where $\mathbb{C}$ denotes the field of complex numbers, with the following properties

$$[\alpha_1 x_1+\alpha_2 x_2, y]=\alpha_1[x_1,y]+\alpha_2[x_2,y],\quad x_1,x_2,y\in\mathcal{X},\ \alpha_1,\alpha_2\in\mathbb{C}, \tag{1.1}$$

and

$$[x,y]=\overline{[y,x]},\quad x,y\in\mathcal{X}, \tag{1.2}$$

that is, it is *linear* with respect to the first variable and *conjugate symmetric*. It follows that the inner product is *conjugate linear* with respect to the second variable, that is,

$$[x, \beta_1 y_1 + \beta_1 y_2] = \overline{\beta}_1 [x, y_1] + \overline{\beta}_2 [x, y_2], \quad x, y_1, y_2 \in \mathcal{X},\ \beta_1, \beta_2 \in \mathbb{C}. \tag{1.3}$$

If the complex vector space $\mathcal{X}$ has a fixed inner product $[\cdot,\cdot]$ then we call it an *inner product space* and use the notation $(\mathcal{X}, [\cdot,\cdot])$. Sometimes, in order to avoid confusion, we use the notation $(\mathcal{X}; [\cdot,\cdot]_{\mathcal{X}})$.

Let us first observe that $[x, x] \in \mathbb{R}$ for all $x \in \mathcal{X}$ and that an inner product $[\cdot,\cdot]$ always satisfies the *polarisation formula*, or *polarisation identity*

$$[x, y] = \frac{1}{4} \sum_{k=0}^{3} \mathrm{i}^k [x + \mathrm{i}^k y, x + \mathrm{i}^k y], \quad x, y \in \mathcal{X}, \tag{1.4}$$

where i denotes the imaginary complex number $\mathrm{i}^2 = -1$. This shows that an inner product $[\cdot,\cdot]$ on $\mathcal{X}$ is fully determined by the associated *quadratic form* $\mathcal{X} \ni x \mapsto [x, x] \in \mathbb{R}$.

A vector $x \in \mathcal{X}$ is *positive, neutral,* or *negative*, by definition, if the corresponding *inner square* $[x, x]$, which is always a real number, is positive $[x, x] > 0$, null $[x, x] = 0$, or negative $[x, x] < 0$, respectively. An inner product space $(\mathcal{X}, [\cdot,\cdot])$ is called *indefinite* if there exist both positive and negative vectors in $\mathcal{X}$, it is called *semidefinite* if it is not indefinite, and it is called *definite* if $\mathcal{X}$ contains no nontrivial neutral vectors. The following result justifies the last definition.

Lemma 1.1.1. *If the inner product space $(\mathcal{X}, [\cdot,\cdot])$ is indefinite then it contains nontrivial neutral vectors.*

Proof. Let $x, y \in \mathcal{X}$ be such that x is positive and y is negative. Since $[x, x] \cdot [y, y] < 0$, there always exists a real solution λ of the equation

$$[x, x] + 2\lambda \operatorname{Re}[x, y] + \lambda^2 [y, y] = 0,$$

hence the vector $z = x + \lambda y$ is neutral. If $z = 0$ then $0 < [x, x] = |\lambda|^2 [y, y] < 0$, a contradiction, hence $z \neq 0$. ∎

So, an inner product space can be, exclusively, either positive semidefinite, or negative semidefinite, or indefinite. On the other hand, a definite inner product space can be, exclusively, either positive definite or negative definite.

For semidefinite inner product spaces we recall the celebrated Schwarz Inequality.

Lemma 1.1.2 (Schwarz Inequality)**.** *If the inner product space $(\mathcal{X}, [\cdot,\cdot])$ is semidefinite then*

$$|[x, y]|^2 \leqslant [x, x] \cdot [y, y], \quad x, y \in \mathcal{X}.$$

Proof. To make a choice, assume that $(\mathcal{X}, [\cdot,\cdot])$ is positive semidefinite. Then, for arbitrary $x, y \in \mathcal{X}$ and all real λ, the vector $z = \lambda x + y$ is nonnegative, hence

$$[z, z] = \lambda^2 [x, x] + 2\lambda \operatorname{Re}[x, y] + [y, y] \geqslant 0.$$

This implies that the discriminant of the second order real polynomial, in the variable λ, from above is nonnegative and hence

$$(\mathrm{Re}[x, y])^2 \leqslant [x, x] \cdot [y, y].$$

Choosing $\theta \in \mathbb{R}$ such that $[x, \mathrm{e}^{\mathrm{i}\theta} y] = \mathrm{Re}[x, y]$, we get the desired inequality.

In the negative semidefinite case, either observe that the proof is similar or consider the positive semidefinite inner product space $(\mathcal{X}; -[\cdot, \cdot])$. ∎

Lemma 1.1.3. *If the inner product space $(\mathcal{X}, [\cdot, \cdot])$ has at least one positive (negative) vector then each vector of $\mathcal{X}$ can be written as a sum of two positive (negative) vectors in $\mathcal{X}$.*

Proof. Let $x_0 \in \mathcal{X}$ be positive and $x \in \mathcal{X}$ arbitrary. Then, for $\lambda > 0$ and sufficiently large, we have

$$[x + \lambda x_0, x + \lambda x_0] = [x, x] + 2\lambda \,\mathrm{Re}[x, x_0] + \lambda^2 [x_0, x_0] > 0,$$

hence $x = x_1 + x_2$, where the vectors $x_1 = x + \lambda x_0$ and $x_2 = -\lambda x_0$ are positive. The statement corresponding to negative vectors has a similar proof. ∎

Throughout this monograph, a subset $\mathcal{A}$ of $\mathcal{X}$ is called a *linear manifold* if it is nonempty and stable under linear operations $\alpha x + \beta y$, for any $\alpha, \beta \in \mathbb{C}$ and any $x, y \in \mathcal{A}$. In order to make things clearer, we reserve the word *subspace* for a linear manifold that is closed, when a certain linear topology on $\mathcal{X}$ is specified.

A subset $\mathcal{A} \subseteq \mathcal{X}$ is called *positive* if $[x, x] \geqslant 0$ for all $x \in \mathcal{A}$ and it is called *strictly positive* if $[x, x] > 0$ for all $x \in \mathcal{A} \setminus \{0\}$. In this respect, a linear manifold $\mathcal{A} \subseteq \mathcal{X}$ is strictly positive if and only if the inner product space $(\mathcal{A}; [\cdot, \cdot]_{\mathcal{A}})$ is positive definite, where $[x, y]_{\mathcal{A}} = [x, y]$ for all $x, y \in \mathcal{A}$. The definitions of *negative* and *strictly negative* sets are now clear. $\mathcal{A}$ is *neutral* if $[x, x] = 0$ for all $x \in \mathcal{A}$.

Since there is no general agreement on the terminology, some remarks are in order. What we call here a positive linear manifold other authors call a *nonnegative* linear manifold and what we call here a strictly positive linear manifold other authors call a *positive* linear manifold, for example. One reason for the terminology that we have adopted in this monograph is that, actually, in agreement with the previous definitions, a positive linear manifold is just an abbreviation for positive semidefinite linear manifold. Another reason for this convention comes from the fact that later we will establish some connections with linear operators for which positive traditionally means positive semidefinite.

Two vectors $x, y \in \mathcal{X}$ are called *orthogonal* if $[x, y] = 0$ and, in this case, we write $x \perp y$. Sometimes, in order to avoid confusion, we may use the more involved notation $[\perp]$, in order to make clear to which inner product the orthogonality is referring. Two subsets $\mathcal{A}, \mathcal{B} \in \mathcal{X}$ are *orthogonal* if $x \perp y$ for all $x \in \mathcal{A}$ and $y \in \mathcal{B}$. The *orthogonal companion* of a subset $\mathcal{A} \subseteq \mathcal{X}$ is, by definition, the linear manifold $\mathcal{A}^\perp \subseteq \mathcal{X}$ defined by

$$\mathcal{A}^\perp = \{x \in \mathcal{X} \mid x \perp y,\ y \in \mathcal{A}\}. \tag{1.5}$$

If $\mathcal{A}$ and $\mathcal{B}$ are nonempty subsets of the vector space $\mathcal{X}$ then we define $\mathcal{A}+\mathcal{B} = \{a+b \mid a \in \mathcal{A},\ b \in \mathcal{B}\}$. If both $\mathcal{A}$ and $\mathcal{B}$ are linear manifolds then $\mathcal{A}+\mathcal{B}$ is a linear manifold as well and coincides with the linear manifold spanned by the vectors in $\mathcal{A} \cup \mathcal{B}$. If $\mathcal{A}$ and $\mathcal{B}$ are linear manifolds and $\mathcal{A} \cap \mathcal{B} = \{0\}$ then we use the notation $\mathcal{A} \dot{+} \mathcal{B}$ for $\mathcal{A}+\mathcal{B}$ and call it a *direct sum*.

If $\mathcal{A} \subseteq \mathcal{B}$ then $\mathcal{B}^\perp \subseteq \mathcal{A}^\perp$. If both subsets $\mathcal{A}$ and $\mathcal{B}$ contain the vector 0 then

$$(\mathcal{A}+\mathcal{B})^\perp = \mathcal{A}^\perp \cap \mathcal{B}^\perp. \tag{1.6}$$

Denoting $\mathcal{A}^{\perp\perp} = (\mathcal{A}^\perp)^\perp$ it follows readily that $\mathcal{A} \subseteq \mathcal{A}^{\perp\perp}$ and $\mathcal{A}^\perp = \mathcal{A}^{\perp\perp\perp}$.

Let $\mathcal{L}$ be a linear manifold in $\mathcal{X}$. The linear manifold $\mathcal{L}^0 = \mathcal{L} \cap \mathcal{L}^\perp$ is called the *isotropic part* of $\mathcal{L}$, and the vectors $x \in \mathcal{L}^0$ are called the *isotropic vectors* of $\mathcal{L}$. We have $\mathcal{X}^0 = \mathcal{X}^\perp$. The linear manifold $\mathcal{L} \subseteq \mathcal{X}$ is called *degenerate* if $\mathcal{L}^0 \neq \{0\}$, it is called *nondegenerate* in the opposite case, and it is called *maximal nondegenerate* if whenever $\mathcal{M}$ is a nondegenerate linear manifold in $\mathcal{X}$ such that $\mathcal{L} \subseteq \mathcal{M}$ it follows that $\mathcal{L} = \mathcal{M}$.

Example 1.1.4. Let $(\omega_n)_{n\geqslant 1}$ be a sequence of real numbers and denote by $\mathcal{X}$ the set of all sequences $(x_n)_{n\geqslant 1}$ of complex numbers such that

$$\sum_{n=1}^{\infty} |\omega_n|\, |x_n|^2 < \infty.$$

Then the formula

$$[x,y] = \sum_{n=1}^{\infty} \omega_n x_n \overline{y}_n, \quad x = (x_n)_{n\geqslant 1},\ y = (y_n)_{n\geqslant 1} \in \mathcal{X},$$

defines an inner product space $(\mathcal{X}, [\cdot,\cdot])$. The inner product space $(\mathcal{X}, [\cdot,\cdot])$ is indefinite if and only if the sequence $(\omega_n)_{n\geqslant 1}$ contains both positive and negative numbers. It is degenerate if and only if the sequence $(\omega_n)_{n\geqslant 1}$ has some null elements. ∎

Remark 1.1.5. Let $(\mathcal{X}, [\cdot,\cdot])$ be an inner product space. Then, a new inner product space can be defined by considering the quotient space $\widehat{\mathcal{X}} = \mathcal{X}/\mathcal{X}^0$ on which a natural inner product is defined

$$[\hat{x}, \hat{y}] = [x,y], \quad x \in \mathcal{X},\ y \in \mathcal{X}.$$

This definition is correct and $(\widehat{\mathcal{X}}, [\cdot,\cdot])$ is nondegenerate. ∎

Here and in the following we use the symbol $\dot{+}$ any time we have a direct sum of two linear manifolds, that is, if $\mathcal{L}_1$ and $\mathcal{L}_2$ are two linear manifolds such that $\mathcal{L}_1 \cap \mathcal{L}_2 = \{0\}$ then $\mathcal{L}_1 \dot{+} \mathcal{L}_2 := \{x_1 + x_2 \mid x_1 \in \mathcal{L}_1,\ x_2 \in \mathcal{L}_2\}$.

Proposition 1.1.6. *Let $(\mathcal{X}; [\cdot,\cdot])$ be an inner product space.* (a) *Every nondegenerate linear manifold of $\mathcal{X}$ is contained in a maximal nondegenerate linear manifold in $\mathcal{X}$.*

(b) *A linear manifold $\mathcal{L}$ of the inner product space $(\mathcal{X}, [\cdot,\cdot])$ is a direct summand of $\mathcal{X}^0$, that is, $\mathcal{X} = \mathcal{L} \dot{+} \mathcal{X}^0$, if and only if it is maximal nondegenerate.*

Proof. (a) This assertion is a consequence of Zorn's Lemma. Briefly, assuming that $\mathcal{N}$ is a nondegenerate linear manifold in $\mathcal{X}$, let $\mathfrak{X}_{\mathcal{N}}$ denote the set of all nondegenerate linear manifolds $\mathcal{L}$ in $\mathcal{X}$ such that $\mathcal{N} \subseteq \mathcal{L}$, and ordered by inclusion. Since $\mathcal{N} \in \mathfrak{X}_{\mathcal{N}}$ is such a linear manifold $\mathfrak{X}_{\mathcal{N}}$ is nonvoid. It is easy to see that any chain in $\mathfrak{X}_{\mathcal{N}}$ has an upper bound, in the sense that, whenever $\{\mathcal{L}_\alpha\}_{\alpha\in A}$ is a family of elements $\mathcal{L}_\alpha \in \mathfrak{X}_{\mathcal{N}}$ indexed by a totally ordered set $(A; \leqslant)$ and such that $\alpha \leqslant \beta$ implies $\mathcal{L}_\alpha \subseteq \mathcal{L}_\beta$, then $\mathcal{L} = \bigcup_{\alpha\in A} \mathcal{L}_\alpha$ is an element in $\mathfrak{X}_{\mathcal{N}}$ such that $\mathcal{L} \supseteq \mathcal{L}_\alpha$ for all $\alpha \in A$. Thus, by Zorn's Lemma, $\mathfrak{X}_{\mathcal{N}}$ has a maximal element $\mathcal{M}$ which turns out to be a maximal nondegenerate linear manifold in $\mathcal{X}$ such that $\mathcal{N} \subseteq \mathcal{M}$.

(b) If $\mathcal{X} = \mathcal{L}\dot{+}\mathcal{X}^0$ holds then $\mathcal{L}$ is nondegenerate and if $\mathcal{M}$ is another linear manifold strictly containing $\mathcal{L}$ then $\mathcal{X}^0 \cap \mathcal{M} \neq \{0\}$, hence $\mathcal{M}$ is degenerate.

Conversely, if $\mathcal{L}$ is maximal nondegenerate then $\mathcal{X}^0 \cap \mathcal{L} = \{0\}$. Let $\mathcal{N}$ be a direct summand of $\mathcal{X}^0\dot{+}\mathcal{L}$, that is, $\mathcal{X} = \mathcal{X}^0\dot{+}\mathcal{L}\dot{+}\mathcal{N}$. Then $\mathcal{L}\dot{+}\mathcal{N}$ is also nondegenerate and, on account of the maximality of $\mathcal{L}$, we get $\mathcal{N} = \{0\}$. ∎

If $\mathcal{L}$ is a linear manifold of the inner product space $(\mathcal{X}; [\cdot,\cdot])$, we denote by $\mathcal{L}_0 = \{x \in \mathcal{L} \mid [x,x] = 0\}$ the *neutral part* of $\mathcal{L}$.

Proposition 1.1.7. *Let $\mathcal{L}$ be a linear manifold in the inner product space $(\mathcal{X}; [\cdot,\cdot])$. Then,*

(a) $\mathcal{L}^0 \subseteq \mathcal{L}_0$.

(b) *The following assertions are equivalent.*

  (i) *$\mathcal{L}_0$ is a linear manifold.*

  (ii) *$\mathcal{L}$ is semidefinite, that is, either positive or negative.*

  (iii) $\mathcal{L}_0 = \mathcal{L}^0$.

(c) *$\mathcal{L}_0 = \{0\}$ if and only if $\mathcal{L}$ is definite, that is, either strictly positive or strictly negative.*

Proof. (a) This is clear, by definition.

(b) (iii)⇒(ii). Assume that $\mathcal{L}$ is not semidefinite, that is, it is indefinite. By Proposition 1.1.6, there exists a maximal nondegenerate linear manifold $\mathcal{M} \subseteq \mathcal{L}$ and hence $\mathcal{L} = \mathcal{L}^0 + \mathcal{M}$, with $\mathcal{L}^0 \cap \mathcal{M} = \{0\}$. Since $\mathcal{L}$ is indefinite, the nondegenerate linear manifold $\mathcal{M}$ is indefinite as well and then, by Lemma 1.1.1, there exists $x \in \mathcal{M} \subseteq \mathcal{L}$ a nontrivial neutral vector, hence $x \in \mathcal{L}_0 \setminus \mathcal{L}^0$.

(ii)⇒(i). This is a consequence of the Schwarz Inequality, see Lemma 1.1.2.

(i)⇒(iiii). If $\mathcal{L}_0 = \mathcal{L}$ then $\mathcal{L}^0 = \mathcal{L}$ and hence $\mathcal{L}_0 = \mathcal{L}^0$ so we can assume, without restricting the generality, that $\mathcal{L}_0 \neq \mathcal{L}$. By (a), we only need to prove $\mathcal{L}_0 \subseteq \mathcal{L}^0$. Let us assume, by contradiction, that there exists $y \in \mathcal{L}_0 \setminus \mathcal{L}^0$, hence $[y,y] = 0$ and there exists $x \in \mathcal{L}$ such that $[x,y] \neq 0$. By changing x with $\mathrm{e}^{\mathrm{i}\theta}x$ for a suitable real number θ we can assume that $[x,y] \in \mathbb{R}$. Then consider the vectors

$$x_\lambda = (1-\lambda)x + \lambda y, \quad \lambda \in \mathbb{C}. \tag{1.7}$$

Depending on the sign of the real number $[x,x]$ we distinguish three possible cases.

If $[x,x]=0$ then, since $\mathcal{L}_0$ is assumed to be a linear manifold and both $x,y\in\mathcal{L}_0$, it follows that $x_\lambda\in\mathcal{L}_0$ for all $\lambda\in\mathbb{C}$, hence, taking into account that $[x,x]=[y,y]=0$ and $[x,y]\in\mathbb{R}$, we have

$$0=[x_\lambda,x_\lambda]=2\operatorname{Re}(1-\lambda)\overline{\lambda}[x,y],\quad \lambda\in\mathbb{C},$$

which implies $[x,y]=0$, a contradiction.

If $[x,x]>0$ then, we consider the real polynomial

$$p(t)=[x_t,x_t]=(1-t)^2[x,x]+2(1-t)t[x,y],\quad t\in\mathbb{R}, \tag{1.8}$$

which has a root for $t_1=1$ and a second root for $t_2=[x,x]/([x,x]-2[x,y])$. Without any loss of generality we can always assume that $[x,x]\neq 2[x,y]$ and, since $[x,y]\neq 0$ it follows that the two roots t_1 and t_2 are distinct. This implies that x_{t_1} and x_{t_2} are neutral vectors hence the whole linear manifold generated by them is neutral, by hypothesis, in particular $x=x_0$ is neutral, a contradiction.

If $[x,x]<0$ then we proceed as in the previous case and get the same contradiction. In conclusion, we have proven that $\mathcal{L}_0=\mathcal{L}^0$.

(c) If $\mathcal{L}_0=\{0\}$ then it is a linear manifold and, by assertion (b), it follows that $\mathcal{L}$ is a nondegenerate semidefinite linear manifold, hence definite. The converse implication is obvious. ∎

In the following we fix an inner product space $(\mathcal{X},[\cdot,\cdot])$. For any linear manifold $\mathcal{L}$ in $\mathcal{X}$ we set $\dim(\mathcal{L})$, the *algebraic dimension* of $\mathcal{L}$, that is, $\dim(\mathcal{L})$ is either 0, if $\mathcal{L}=\{0\}$, or a natural number n, if n is the maximal number of linearly independent vectors in $\mathcal{L}$, or the symbol ∞, if $\mathcal{L}$ has linearly independent subsets of vectors of any finite cardinality.

Lemma 1.1.8. *For any linear manifolds $\mathcal{L}$ and $\mathcal{M}$ of $\mathcal{X}$ we have*

$$\dim(\mathcal{L}\cap\mathcal{M}^\perp)+\dim(\mathcal{M})\geqslant\dim(\mathcal{L}).$$

Proof. If either $\dim(\mathcal{M})=\infty$ or $\dim(\mathcal{L})\leqslant\dim(\mathcal{M})$ holds, then the inequlity is true. Let $\dim(\mathcal{M})=m<\infty$ and assume that for $l>m$ there exists a linearly independent system of vectors $\{e_j\}_{j=1}^l$ in $\mathcal{L}$. Let $\{f_k\}_{k=1}^m$ be a basis for $\mathcal{M}$. Then there exist at least $l-m$ linearly independent vectors of the form $x=\sum_{j=1}^l\alpha_je_j$ such that

$$\sum_{j=1}^l\alpha_j[e_j,f_k]=0,\quad k=1,\dots,m,$$

equivalently, $x\in\mathcal{L}\cap\mathcal{M}^\perp$, and hence $\dim(\mathcal{L}\cap\mathcal{M}^\perp)\geqslant l-m$. ∎

Corollary 1.1.9. *If $\mathcal{L}\cap\mathcal{M}^\perp=\{0\}$ then $\dim(\mathcal{L})\leqslant\dim(\mathcal{M})$.*

By definition, the *algebraic ranks of negativity, isotropy* and *positivity* of $\mathcal{X}$ are, respectively,

$$\begin{aligned} \kappa_-(\mathcal{X}) &= \sup\{\dim(\mathcal{L}) \mid \mathcal{L} \text{ is a strictly negative linear manifold in } \mathcal{X}\}, \\ \kappa_0(\mathcal{X}) &= \dim(\mathcal{X}^0), \\ \kappa_+(\mathcal{X}) &= \sup\{\dim(\mathcal{L}) \mid \mathcal{L} \text{ is a strictly positive linear manifold in } \mathcal{X}\}. \end{aligned} \tag{1.9}$$

Clearly, these ranks are either natural numbers or the symbol ∞. Due to the polarisation formula (1.4), these ranks are also called the *number of negative (null, positive) squares* of the associated quadratic form $\mathcal{X} \ni x \mapsto [x, x]$. In addition, the *rank of indefiniteness* of an inner product space $(\mathcal{X}; [\cdot, \cdot])$ is, by definition, $\kappa(\mathcal{X}) = \min\{\kappa_-(\mathcal{X}), \kappa_+(\mathcal{X})\}$. The triple $(\kappa_-(\mathcal{X}), \kappa_0(\mathcal{X}), \kappa_+(\mathcal{X}))$ is also called the *inertia* of the inner product space $(\mathcal{X}; [\cdot, \cdot])$.

Example 1.1.10. Let $(\mathcal{H}, \langle\cdot, \cdot\rangle)$ be a Hilbert space and let $A \in \mathcal{B}(\mathcal{H})$ be selfadjoint. Recall that, for a Hilbert space $\mathcal{H}$, we denote by $\mathcal{B}(\mathcal{H})$ the collection of all linear bounded operators $T\colon \mathcal{H} \to \mathcal{H}$. Set $\mathcal{X}_A = \mathcal{H}$, define an inner product $[\cdot, \cdot]_A$ by

$$[x, y]_A = \langle Ax, y\rangle, \quad x, y \in \mathcal{X}_A,$$

and consider the inner product space $(\mathcal{X}_A, [\cdot, \cdot]_A)$. Then $\kappa_-(\mathcal{X}_A)$ $(\kappa_+(\mathcal{X}_A))$ is equal to the algebraic dimension of the spectral subspace of A corresponding to the negative (positive) semi-axis and $\kappa_0(\mathcal{X}_A) = \dim(\mathrm{Ker}(A))$. If $\kappa_-(\mathcal{X}_A) < \infty$ $(\kappa_+(\mathcal{X}_A) < \infty)$ then this number is equal to the number of negative (positive) eigenvalues of A, counted with their multiplicities. ∎

Let $(\mathcal{X}; [\cdot, \cdot]_{\mathcal{X}})$ and $(\mathcal{Y}; [\cdot, \cdot]_{\mathcal{X}})$ be two inner product spaces. A linear mapping $T\colon \mathcal{X} \to \mathcal{Y}$ is called *isometric* if

$$[Tx_1, Tx_2]_{\mathcal{Y}} = [x_1, x_2]_{\mathcal{X}}, \quad x_1, x_2 \in \mathcal{X}.$$

If, in addition, the linear isometric operator T is a bijection, then we call it an *isomorphism* of inner product spaces.

Remark 1.1.11. Let $T\colon \mathcal{X} \to \mathcal{Y}$ be linear and isometric. Then for an arbitrary strictly positive linear manifold $\mathcal{L} \subseteq \mathcal{X}$ the mapping $T|\mathcal{L}$ is injective. This shows that $\kappa_+(\mathcal{X}) \leqslant \kappa_+(\mathcal{Y})$. Similarly we have $\kappa_-(\mathcal{X}) \leqslant \kappa_-(\mathcal{Y})$. In particular, if an isometric linear bijection $T\colon \mathcal{X} \to \mathcal{Y}$ exists then $\kappa_\pm(\mathcal{X}) = \kappa_\pm(\mathcal{Y})$ and, since $T(\mathcal{X}^0) = \mathcal{Y}^0$ we also have $\kappa_0(\mathcal{X}) = \kappa_0(\mathcal{Y})$. Consequently, isomorphisms of inner product spaces preserve the inertia. ∎

Proposition 1.1.12. *Given an inner product space $(\mathcal{X}; [\cdot, \cdot]_{\mathcal{X}})$, $\kappa_-(\mathcal{X})$ $(\kappa_+(\mathcal{X}))$ is equal to the supremum of the number of negative (positive) eigenvalues, counted with their multiplicities, of the matrices $([x_i, x_j])_{i,j=1}^n$, where $n \geqslant 1$ and $\{x_i\}_{i=1}^n \subset \mathcal{X}$.*

Proof. Let $\mathcal{L}$ be a strictly negative (strictly positive) linear manifold of $\mathcal{X}$ of dimension $m < \infty$. By the Gram–Schmidt orthogonalisation procedure we obtain a system of vectors $\{e_i\}_{i=1}^m$ of $\mathcal{L}$ such that $[e_i, e_j] = -\delta_{ij}$ ($[e_i, e_j] = \delta_{ij}$), $i, j = 1, \ldots, m$. Hence the matrix $([e_i, e_j])_{i,j=1}^m$ has exactly m negative (positive) eigenvalues, counted with their multiplicities.

Conversely, let $\{x_i\}_{i=1}^n$ be a finite system of vectors of $\mathcal{X}$. Without restricting the generality, we can assume that this system of vectors is linearly independent. If $\mathcal{L}$ denotes the linear span of $\{x_i\}_{i=1}^n$ then $\dim(\mathcal{L}) = n$. Consider the matrix $A = ([x_i, x_j])_{i,j=1}^n$ as a selfadjoint operator on the Hilbert space $\mathbb{C}^n$. With notation as in Example 1.1.10, we have

$$[x, y] = [\alpha, \beta]_A, \quad x, y \in \mathcal{L}, \tag{1.10}$$

where $x = \sum_{i=1}^n \alpha_i x_i$, $y = \sum_{j=1}^n \beta_j x_j$, $\alpha = (\alpha_1, \ldots, \alpha_n) \in \mathbb{C}^n$, and $\beta = (\beta_1, \ldots, \beta_n) \in \mathbb{C}^n$. The mapping

$$\mathcal{L} \ni x = \sum_{i=1}^n \alpha_i x_i \mapsto \alpha = (\alpha_1, \ldots, \alpha_n) \in \mathcal{X}_A,$$

is a linear bijection of $\mathcal{L}$ onto $\mathcal{X}_A$ and (1.10) shows that it preserves the inner products. Hence $\kappa_-(\mathcal{L})$ ($\kappa_+(\mathcal{L})$) coincides with the number of negative (positive) eigenvalues of the matrix A, counted with their multiplicities. ∎

An inner product space $(\mathcal{X}; [\cdot, \cdot])$ is called *decomposable* if there exist $\mathcal{L}$ a strictly positive subspace of $\mathcal{X}$ and $\mathcal{M}$ a strictly negative subspace of $\mathcal{X}$ such that $\mathcal{L} \perp \mathcal{M}$ and

$$\mathcal{X} = \mathcal{L} + \mathcal{X}^0 + \mathcal{M}. \tag{1.11}$$

Clearly, in this case we have $\mathcal{X} = \mathcal{L} \dot{+} \mathcal{X}^0 \dot{+} \mathcal{M}$. In this case, a decomposition as in (1.11) is called a *fundamental decomposition* .

A positive linear manifold $\mathcal{L} \in \mathcal{X}$ is called *maximal positive* if there exists no positive linear manifold $\mathcal{L}' \in \mathcal{X}$, different from $\mathcal{L}$, such that $\mathcal{L} \subset \mathcal{L}'$. Similarly one can define *maximal strictly positive, maximal negative*, and *maximal strictly negative* linear manifolds.

Remark 1.1.13. It is easy to see that, in a decomposition as in (1.11), the sum is direct, $\mathcal{L}^\perp = \mathcal{X}^0 + \mathcal{M}$, and $\mathcal{M}^\perp = \mathcal{X}^0 + \mathcal{L}$. Also, the subspace $\mathcal{L}$ is maximal strictly positive and the subspace $\mathcal{M}$ is maximal strictly negative. ∎

Proposition 1.1.14. *In any inner product space, all maximal positive linear manifolds have the same dimension. The same is true for all maximal strictly positive linear manifolds, all maximal negative linear manifolds, and all maximal strictly negative linear manifolds, respectively.*

Proof. Let $\mathcal{L}$ be a maximal strictly positive linear manifold of the inner product space $\mathcal{X}$. Then $\mathcal{L}^\perp$ is a negative linear manifold and hence, if $\mathcal{M}$ is another maximal strictly positive linear manifold, then $\mathcal{M} \cap \mathcal{L}^\perp = \{0\}$ and then, by Lemma 1.1.8, it follows that $\dim(\mathcal{L}) \leqslant \dim(\mathcal{M})$. By symmetry the converse inequality must hold as well, hence

$\dim(\mathcal{L}) = \dim(\mathcal{M})$. For maximal strictly negative linear manifolds the reasoning is similar.

Let now $\mathcal{L}$ and $\mathcal{M}$ be two maximal positive linear manifolds of $\mathcal{X}$. Then $\mathcal{M}^\perp$ is a negative linear manifold and hence $\mathcal{L} \cap \mathcal{M}^\perp$ is neutral, hence $\mathcal{M} + \mathcal{L} \cap \mathcal{M}^\perp$ is a positive linear manifold and, since $\mathcal{M}$ is maximal positive, it follows that $\mathcal{L} \cap \mathcal{M}^\perp \subseteq \mathcal{M}$. In view of Lemma 1.1.8 it follows that $\mathcal{L}$ and $\mathcal{M}$ are simultaneously infinite dimensional.

It remains to investigate the case when both $\mathcal{L}$ and $\mathcal{M}$ are finite dimensional. In this case, in view of maximality we have $\mathcal{X}^\circ \subseteq \mathcal{L}, \mathcal{M}$. Hence $\mathcal{X}^\circ$ is finite dimensional as well and, by factoring out $\mathcal{X}^\circ$, without loss of generality we can assume that $\mathcal{X}^\circ = \{0\}$. In addition, in view of Proposition 1.1.12, it follows that $\kappa_+(\mathcal{X})$ is finite. Then $\mathcal{X} = \mathcal{X}_- \dot{+} \mathcal{X}_+$, where $\mathcal{X}_+$ is a finite dimensional maximal strictly positive linear manifold and $\mathcal{X}_-$ is a maximal strictly negative linear manifold. The proof will be complete if we show that $\dim(\mathcal{L}) = \kappa_+(\mathcal{X}) = \dim(\mathcal{X}_+)$.

To see this, in view of the decomposition $\mathcal{X} = \mathcal{X}_- \dot{+} \mathcal{X}_+$, each vector $x \in \mathcal{L}$ is uniquely decomposed as $x = x_+ + x_-$, with $x_+ \in \mathcal{X}_+$ and $x_- \in \mathcal{X}_-$. Since $\mathcal{L}$ is positive, we have $0 \leqslant [x, x] = [x_+, x_+] + [x_-, x_-]$, hence $[x_+, x_+] \geqslant -[x_-, x_-]$ and, consequently, the mapping $\mathcal{L} \ni x \mapsto x_+ \in \mathcal{X}_+$ is injective and, in view of the maximality of $\mathcal{L}$, it is surjective as well. This shows that $\dim(\mathcal{L}) = \dim(\mathcal{X}_+)$. ∎

A linear manifold $\mathcal{L}$ of an inner product space $(\mathcal{X}; [\cdot, \cdot])$ is called *orthocomplemented* if there exists a linear manifold $\mathcal{M}$ in $\mathcal{X}$ such $\mathcal{L} \perp \mathcal{M}$, $\mathcal{M} \cap \mathcal{L} = \{0\}$ and $\mathcal{X} = \mathcal{L} + \mathcal{M}$. The linear manifold $\mathcal{M}$ with this property is called an *orthocomplement* of $\mathcal{L}$.

Remark 1.1.15. We fix an inner product space $(\mathcal{X}; [\cdot, \cdot])$ and a linear manifold $\mathcal{L}$ in $\mathcal{X}$.

(a) $\mathcal{L}$ is orthocomplemented in $\mathcal{X}$ if and only if there exists a linear operator $P \colon \mathcal{X} \to \mathcal{X}$ such that it is *idempotent*, that is, $P^2 = P$, *Hermitian*, equivalently , that is, $[Px, y] = [x, Py]$ for all $x, y \in \mathcal{X}$, and $\operatorname{Ran}(P) = \mathcal{L}$. The operator P is uniquely determined by $\mathcal{L}$ and $\mathcal{M}$ and it is called the *orthogonal projection* on $\mathcal{L}$ along $\mathcal{M}$. More precisely, if $\mathcal{M}$ denotes an orthocomplement of $\mathcal{L}$, then for any $x \in \mathcal{X}$ there exist unique $x_1 \in \mathcal{L}$ and $x_2 \in \mathcal{M}$ such that $x = x_1 + x_2$ and we let $Px = x_1$. In this case, $\operatorname{Ker}(P) = \mathcal{M}$ and $Q = I - P$ is the orthogonal projection on $\mathcal{M}$.

(b) If $\mathcal{L}$ is orthocomplemented in $\mathcal{X}$ and $\mathcal{M}$ is its orthocomplement then $\mathcal{X}^0 = \mathcal{L}^0 \dot{+} \mathcal{M}^0$. In particular, this shows that, in general, we do not have uniqueness for the orthocomplement and hence we do not have uniqueness for the orthogonal projection.

(c) Assume that $\mathcal{X}$ is nondegenerate. If $\mathcal{L}$ is orthocomplemented then, from (b) it follows that $\mathcal{L}$ and any of its orthocomplements are nondegenerate. Also, in this case $\mathcal{L}^\perp$ is the unique orthocomplement of $\mathcal{L}$ and the orthogonal projection on $\mathcal{L}$ is unique as well.

(d) Assume that $\mathcal{L}$ is nondegenerate and that $\kappa_\pm(\mathcal{L}) < \infty$. Then $\mathcal{L}$ is orthocomplemented. Indeed, since $\kappa = \dim(\mathcal{L}) = \kappa_-(\mathcal{L}) + \kappa_+(\mathcal{L}) < \infty$, by the Gram–Schmidt orthonormalisation procedure there exists an *orthonormal basis* $\{e_j\}_{j=1}^\kappa$ of $\mathcal{L}$, that is, $[e_j, e_k] = \pm\delta_{j,k}$ for all $j, k \in \{1, \dots, \kappa\}$. Let $P \colon \mathcal{X} \to \mathcal{X}$ be defined by $Px = \sum_{j=1}^\kappa [x, e_j] e_j$, for all $x \in \mathcal{X}$. Then P is an orthogonal projection with $\operatorname{Ran}(P) = \mathcal{L}$ and then we apply the remark at item (a). ∎

Proposition 1.1.16. *Let $\mathcal{L}$ be a strictly positive (strictly negative) linear manifold in the inner product space $(\mathcal{X};[\cdot,\cdot])$. The following assertions are equivalent.*

(i) *$\mathcal{X}$ has a fundamental decomposition $\mathcal{X}=\mathcal{L}+\mathcal{X}^0+\mathcal{M}$.*
(ii) *$\mathcal{L}$ is maximal strictly positive and orthocomplemented.*

Proof. (i)⇒(ii). To make a choice, assume that $\mathcal{L}$ is strictly positive. If $\mathcal{X}$ has a fundamental decomposition $\mathcal{X}=\mathcal{L}+\mathcal{X}^0+\mathcal{M}$ then $\mathcal{X}^0+\mathcal{M}$ is a negative linear manifold and an orthocomplement of $\mathcal{L}$. Then $\mathcal{L}$ is a maximal strictly positive linear manifold.

(ii)⇒(i). If $\mathcal{L}$ is an orthocomplemented maximal strictly positive linear manifold of $\mathcal{X}$ then $\mathcal{X}=\mathcal{L}+\mathcal{L}^\perp$ and $\mathcal{L}^\perp$ is a negative linear manifold. By Proposition 1.1.6 it follows that $\mathcal{L}^\perp=(\mathcal{L}^\perp)^0+\mathcal{M}$, for some strictly negative linear manifold $\mathcal{M}$. It is easy to see that $(\mathcal{L}^\perp)^0=\mathcal{X}^0$ and that $\mathcal{X}=\mathcal{L}+\mathcal{X}^0+\mathcal{M}$, hence $\mathcal{X}$ has a fundamental decomposition of the required type. ∎

Proposition 1.1.17. *Let $(\mathcal{X};[\cdot,\cdot])$ be an inner product space with finite rank of indefiniteness $\kappa(\mathcal{X})=\min\{\kappa_+(\mathcal{X}),\kappa_-(\mathcal{X})\}<\infty$. Then it is decomposable.*

Proof. To make a choice, let us assume that $\kappa_-(\mathcal{X})<\infty$ and let $\mathcal{L}$ be a maximal strictly negative linear manifold of $\mathcal{X}$. Then, by Proposition 1.1.14 we have $\dim(\mathcal{L})=\kappa_-(\mathcal{X})$ and, by Remark 1.1.15.(d), $\mathcal{L}$ is orthocomplemented, hence Proposition 1.1.16 shows that $\mathcal{X}$ is decomposable. ∎

1.2 The Weak Topology

Let $(\mathcal{X},[\cdot,\cdot])$ be an inner product space. The locally convex topology on $\mathcal{X}$ defined by the family of seminorms $\{p_y\}_{y\in\mathcal{X}}$

$$p_y(x)=|[x,y]|,\quad x\in\mathcal{X},$$

is called *the weak topology*. Since

$$\mathcal{X}^0=\bigcap_{y\in\mathcal{X}}\operatorname{Ker}(p_y), \tag{2.1}$$

it follows that the weak topology is separated if and only if the inner product space $(\mathcal{X},[\cdot,\cdot])$ is nondegenerate. Also, the inner product $[\cdot,\cdot]$ is always *separately continuous*, that is, the linear functionals $\mathcal{X}\ni x\mapsto[x,y]$ are continuous for all $y\in\mathcal{X}$, with respect to the weak topology.

Theorem 1.2.1. *A linear functional φ on the inner product space $\mathcal{X}$ is weakly continuous if and only if there exists a vector $y_0\in\mathcal{X}$ such that*

$$\varphi(x)=[x,y_0],\quad x\in\mathcal{X}.$$