

# Contents

	<i>Introduction</i>	page ix
	<i>Before we Begin</i>	xiii
<b>Part I</b>	<b>Rings</b>	1
<b>1</b>	<b>The Integers</b>	3
	1.1 The Well-Ordering Principle and Induction	3
	1.2 ‘Division with Remainder’ in $\mathbb{Z}$	6
	1.3 Greatest Common Divisors	7
	1.4 The Fundamental Theorem of Arithmetic	12
	Exercises	20
<b>2</b>	<b>Modular Arithmetic</b>	22
	2.1 Equivalence Relations and Quotients	22
	2.2 Congruence mod $n$	23
	2.3 Algebra in $\mathbb{Z}/n\mathbb{Z}$	26
	2.4 Properties of the Operations $+$ , $\cdot$ on $\mathbb{Z}/n\mathbb{Z}$	29
	2.5 Fermat’s Little Theorem, and the RSA Encryption System	34
	Exercises	39
<b>3</b>	<b>Rings</b>	41
	3.1 Definition and Examples	41
	3.2 Basic Properties	47
	3.3 Special Types of Rings	51
	Exercises	57
<b>4</b>	<b>The Category of Rings</b>	59
	4.1 Cartesian Products	59
	4.2 Subrings	61
	4.3 Ring Homomorphisms	64
	4.4 Isomorphisms of Rings	69
	Exercises	75

vi	<b>Contents</b>	
<b>5</b>	<b>Canonical Decomposition, Quotients, and Isomorphism Theorems</b>	77
	5.1 Rings: Canonical Decomposition, I	77
	5.2 Kernels and Ideals	79
	5.3 Quotient Rings	83
	5.4 Rings: Canonical Decomposition, II	89
	5.5 The First Isomorphism Theorem	91
	5.6 The Chinese Remainder Theorem	93
	5.7 The Third Isomorphism Theorem	100
	Exercises	103
<b>6</b>	<b>Integral Domains</b>	106
	6.1 Prime and Maximal Ideals	106
	6.2 Primes and Irreducibles	110
	6.3 Euclidean Domains and PIDs	112
	6.4 PIDs and UFDs	117
	6.5 The Field of Fractions of an Integral Domain	120
	Exercises	126
<b>7</b>	<b>Polynomial Rings and Factorization</b>	129
	7.1 Fermat's Last Theorem for Polynomials	129
	7.2 The Polynomial Ring with Coefficients in a Field	131
	7.3 Irreducibility in Polynomial Rings	136
	7.4 Irreducibility in $\mathbb{Q}[x]$ and $\mathbb{Z}[x]$	139
	7.5 Irreducibility Tests in $\mathbb{Z}[x]$	143
	Exercises	148
	<b>Part II Modules</b>	151
<b>8</b>	<b>Modules and Abelian Groups</b>	153
	8.1 Vector Spaces and Ideals, Revisited	153
	8.2 The Category of $R$ -Modules	158
	8.3 Submodules, Direct Sums	161
	8.4 Canonical Decomposition and Quotients	164
	8.5 Isomorphism Theorems	168
	Exercises	171
<b>9</b>	<b>Modules over Integral Domains</b>	174
	9.1 Free Modules	174
	9.2 Modules from Matrices	181
	9.3 Finitely Generated vs. Finitely Presented	187
	9.4 Vector Spaces are Free Modules	190
	9.5 Finitely Generated Modules over Euclidean Domains	193
	9.6 Linear Transformations and Modules over $k[t]$	197
	Exercises	199

<b>10</b>	<b>Abelian Groups</b>	202
	10.1 The Category of Abelian Groups	202
	10.2 Cyclic Groups, and Orders of Elements	207
	10.3 The Classification Theorem	212
	10.4 Fermat's Theorem on Sums of Squares	217
	Exercises	223
<b>Part III</b>	<b>Groups</b>	227
<b>11</b>	<b>Groups—Preliminaries</b>	229
	11.1 Groups and their Category	229
	11.2 Why Groups? Actions of a Group	235
	11.3 Cyclic, Dihedral, Symmetric, Free Groups	241
	11.4 Canonical Decomposition, Normality, and Quotients	253
	11.5 Isomorphism Theorems	260
	Exercises	264
<b>12</b>	<b>Basic Results on Finite Groups</b>	267
	12.1 The Index of a Subgroup, and Lagrange's Theorem	267
	12.2 Stabilizers and the Class Equation	269
	12.3 Classification and Simplicity	274
	12.4 Sylow's Theorems: Statements, Applications	276
	12.5 Sylow's Theorems: Proofs	280
	12.6 Simplicity of $\mathcal{A}_n$	282
	12.7 Solvable Groups	286
	Exercises	289
<b>Part IV</b>	<b>Fields</b>	293
<b>13</b>	<b>Field Extensions</b>	295
	13.1 Fields and Homomorphisms of Fields	295
	13.2 Finite Extensions and the Degree of an Extension	299
	13.3 Simple Extensions	301
	13.4 Algebraic Extensions	307
	13.5 Application: 'Geometric Impossibilities'	309
	Exercises	314
<b>14</b>	<b>Normal and Separable Extensions, and Splitting Fields</b>	317
	14.1 Simple Extensions, Again	317
	14.2 Splitting Fields	320
	14.3 Normal Extensions	326
	14.4 Separable Extensions; and Simple Extensions Once Again	328
	14.5 Application: Finite Fields	333
	Exercises	337

viii	<b>Contents</b>	
<b>15</b>	<b>Galois Theory</b>	340
	15.1 Galois Groups and Galois Extensions	340
	15.2 Characterization of Galois Extensions	345
	15.3 The Fundamental Theorem of Galois Theory	350
	15.4 Galois Groups of Polynomials	357
	15.5 Solving Polynomial Equations by Radicals	361
	15.6 Other Applications	370
	Exercises	373
	<b>Appendix A Background</b>	377
	<b>Appendix B Solutions to Selected Exercises</b>	402
	<i>Index of Definitions</i>	461
	<i>Index of Theorems</i>	463
	<i>Subject Index</i>	465