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# Invariance of Modules under Automorphisms of their Envelopes and Covers

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# Contents

# Preface

The study of modules that are invariant under the action of certain subsets of the endomorphism ring of their injective envelope can be drawn back to the pioneering work of Johnson and Wong in which they characterized quasi-injective modules as those modules that are invariant under any endomorphism of their injective envelope. Later, Dickson and Fuller studied modules that are invariant under the group of all automorphisms of their injective envelope and proved that any indecomposable automorphism-invariant module over an $\mathbb{F}$-algebra $A$ is quasi-injective, provided that $\mathbb{F}$ is a field with more than two elements. But after that, this topic remained in dormant stage for some time until Lee and Zhou picked it up again in their paper where they called such modules auto-invariant modules. But the major breakthrough on this topic came from two papers that appeared a few months later. One of them was a paper of Er, Singh and Srivastava, where they proved that the automorphism-invariant modules are precisely the pseudo-injective modules studied earlier by Teply, Jain, Clark, Huynh and others. The other one was a paper by Guil Asensio and Srivastava, where they proved that automorphism-invariant modules satisfy the exchange property and also that they provide a new class of clean modules. Soon after this, Guil Asensio and Srivastava extended the result of Dickson and Fuller by proving that if $A$ is an algebra over a field $\mathbb{F}$ with more than two elements, then a module over $A$ is automorphism invariant if and only if it is quasi-injective.

In 2015, in a paper published in the *Israel Journal of Mathematics*, Guil Asensio, Tutuncu and Srivastava laid down the foundation of general theory of modules invariant under automorphisms (respectively endomorphisms) of envelopes and covers. In this general theory of modules invariant under automorphisms (respectively endomorphisms) of envelopes and covers, we have obtained many interesting properties of such modules and found examples

of some important classes of modules. When this theory is applied to some particular situations, then we obtain results that extend and simplify several results existing in the literature. For example, as a consequence of these general results, one obtains that modules invariant under automorphisms of their injective (respectively pure-injective) envelopes satisfy the full exchange property. These results extend well-known results of Warfield, Fuchs, Huisgen-Zimmermann and Zimmermann. Most importantly, this study yields us a new tool and new perspective to look at generalizations of injective, pure-injective or flat-cotorsion modules. Until now, most of the generalizations of injective modules were focussed on relaxing conditions on lifting of homomorphisms, but this theory has opened up a whole new direction in the study of module theory.

Chapter 1 presents basic definitions and results in ring theory and module theory that is needed to understand the content of this monograph. In Chapter 2, we give the basics of the general theory of envelopes and covers and introduce the theory of modules invariant under automorphisms of their envelopes. In Chapter 1, we also show the connection of this theory to the additive unit structure of elements in von Neumann regular rings. Chapter 3 presents the decomposition theorem of modules invariant under automorphisms of their envelopes and the structure of endomorphism rings of such modules. In Chapter 4, we discuss the particular case of modules invariant under automorphisms of their injective envelopes. In Chapter 5, we present the dual theory of modules coinvariant under automorphisms of their covers.

In Chapter 6, we study the Schröder–Bernstein problem for modules invariant under automorphism of their envelopes and covers. The Schröder–Bernstein theorem is a classical result in basic set theory. The type of problem where one asks if two mathematical objects $A$ and $B$ that are similar in some sense to part of each other are also similar is usually called the Schröder–Bernstein problem. The question whether Schröder–Bernstein property holds for automorphism-invariant modules was raised by Facchini in one of his papers that was later answered in the affirmative by Guil Asensio, Kaleboğaz and Srivastava. Recently, Tuganbaev developed the theory of automorphism-extendable modules; and, along with his coauthors Abyzov, Quynh and others, he has also developed the theory of automorphism-liftable modules. Chapter 7 presents the theory of automorphism-extendable modules, and Chapter 8 presents the theory of automorphism-liftable modules. We conclude with Chapter 9, which presents a list of open problems.

In the past seven years, there has been a lot of activity on these topics, and about a hundred papers have appeared on the topic. The topics covered in this

monograph reflect the personal preferences of authors, and we do not claim to provide an exhaustive survey of the area. We sincerely believe that the tools developed in this monograph will help researchers to employ new techniques in solving various long-standing open problems in the theory of modules.

Pedro A. Guil Asensio would like to dedicate this monograph to his partner Rosa. The work of Pedro A. Guil has been partially supported by the Spanish Government under grant number MTM2016-77445-P, which includes FEDER funds of the EU, and by Fundación Séneca of Murcia under grant number 19880/GERM/15.

Ashish K. Srivastava would like to dedicate this monograph to the loving memory of his mother Giriza Srivastava and his son Anupam Srivastava, a budding astronomer who left Planet Earth too soon. He would like to thank his daughter Ananya and son Arnab for giving him a reason to live and smile, and he would also like to thank his father Suresh C. Srivastava; wife Sweta; siblings Ashutosh and Amitabh; Amrita and other family members and friends for their support and encouragement. The work of Ashish K. Srivastava is partially supported by a grant from Simons Foundation (grant number 426367).

Askar Tuganbaev would like to dedicate this monograph to his wife Natalie, his sister Saule, and his children and grandchildren Saule, Askar, Diar, Timur, Arman, Maxim, Ivar, Polina, Anvar, Dina and Arthur. The work of Askar Tuganbaev is supported by Russian Scientific Foundation, project 16-11-10013P.