1
Preliminaries

In this chapter, we review what will be needed to understand the content of this monograph. Most of the basic facts will be stated without proof.

1.1 Basics of Ring Theory and Module Theory

We begin by recalling basic definitions from ring theory and module theory. An additive abelian group R with addition $+$ is called a ring if R is also a multiplicative monoid with respect to one more operation of multiplication and

$$x(y+z) = xy + xz, \quad (y+z)x = yx + zx \quad \text{for all} \quad x, y, z \in R.$$

All our rings are assumed to contain a nonzero identity element. For a ring R, the center $Z(R)$ is the center of the multiplicative monoid $(R, \cdot)$, i.e. $Z(R) = \{c \in R \mid ca = ac \text{ for all } a \in R\}$, and $Z(R)$ contains the zero element 0_R of the ring R. If R is a ring and there exists a positive integer $n \in \mathbb{N}$ such that $na = 0$ for all $a \in R$, then the least positive integer n with this property is called the characteristic of R; it is denoted by char R. By our definition, any ring R contains at least two elements: the zero element 0_R and the identity element $1_R \neq 0_R$ contained in the center $Z(R)$ of R.

A ring R is called a division ring if every nonzero element of R is invertible (i.e. the set of all nonzero elements of R is a group with respect to multiplication). A commutative division ring is called a field. An element e of the semigroup X is called an idempotent if $e * e = e$. If e is a nonzero idempotent in R, then eRe is a subring of R with identity element e, and eRe is not a unitary subring of R for $e \neq 1$. The idempotents 0 and 1 of the ring R are central in R; they are called trivial idempotents, and the remaining idempotents of R are called nontrivial. If $\{e_i\}_{i \in I}$ is some set of idempotents in the ring R

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and $e_ie_j = e_je_i = 0$ for all $i \neq j$, then e_i are called orthogonal idempotents. If $e_1, \dots, e_n$ are orthogonal idempotents of R and $e_1 + \cdots + e_n = 1_R$, then $\{e_i\}_{i=1}^n$ is called a complete set of orthogonal idempotents of R. For any (resp., central) idempotent $e \in R$, the set $\{1, 1-e\}$ is a complete set of (resp., central) orthogonal idempotents.

Let R be a ring, and let M be an additive abelian group. The group M is called a right (resp., left) R-module if, for any $m \in M$ and $a \in R$, the element ma in M (resp., am in M) is uniquely defined, $(x+y)a = xa + ya$ (resp., $a(x+y) = ax + ay$), $x1 = x$ (resp., $1x = x$), and $x(ab) = (xa)b$ (resp., $(ba)x = b(ax)$) for all $x, y \in M$ and $a, b \in R$. We denote by M_R (resp., $_RM$) the property that M is a right (resp., left) R-module.

If M is a right (resp., left) R-module and X is a subgroup of the additive group M such that $xa \in X$ (resp., $ax \in X$) for all $x \in X$ and $a \in R$, then X is called a submodule of M. A submodule X of M is said to be proper if $X \neq M$. A subgroup I of the additive group $(R, +)$ of the ring R is called a right (resp., left) ideal of R if $xr \in I$ (resp., $rx \in I$) for all $r \in R$ and $x \in I$. Thus, right (resp., left) ideals of R coincide with submodules of the module R_R (resp., $_RR$). An ideal or a two-sided ideal of R is a subset of R that is both a left and a right ideal.

Let A, B be two rings, and let M be a left A-module that also is a right B-module and $(am)b = a(mb)$ for all $m \in M$, $a \in A$, and $b \in B$. Then M is called an A-B-bimodule. We denote by $_AM_B$ the property that $_AM_B$ is an A-B-bimodule. If X is a subset of $_AM_B$ such that $ax \in X$ and $xb \in X$ for all $x \in X$, $a \in A$, and $b \in B$, then X is called a sub-bimodule of $_AM_B$.

If X is a subset of a right (resp., left) R-module M, then we denote by XR (resp., RX) the submodule of M consisting of all finite sums $\sum x_ia_i$ (resp., $\sum a_ix_i$) in X, where $x_i \in X$ and $a_i \in R$. Thus, for any subset X of the right R-module (resp., left R-module) M, we have the submodule XR (resp., RX); if X consists of a single element x, this submodule is called a cyclic module with generator x. Thus, for any subset B of R, we have the right ideal BR and the left ideal RB; if B consists of a single element b, this right (resp., left) ideal is called a principal right (resp., left) ideal with generator x.

The ring R is an (R_1, R_2)-bimodule for any unitary subrings R_1, R_2 of R. In particular, the ring R is an (R, R)-bimodule, and its sub-bimodules coincide with ideals of R. Every additive abelian group M is turned into a module over the ring of integers $\mathbb{Z}$ if for any $m \in M$ and $n \in \mathbb{N}$, we assume that

$$xn = \underbrace{x + \cdots + x}_{n \text{ times}}, \quad x(-n) = -xn, \quad x0_{\mathbb{Z}} = 0_M.$$

Thus, abelian groups coincide with $\mathbb{Z}$-modules.

Let X and Y be two additive abelian groups. A mapping $f: X \to Y$ is called a group homomorphism if $f(x+y) = f(x) + f(y)$, $f(-x) = -f(x)$ and $f(0_X) = 0_Y$ for all $x, y \in X$. The subset $\{f(x) \mid x \in X\}$ in Y is called the image of the homomorphism f; it is denoted by $f(X)$ or $\mathrm{Im}(f)$. The subset $\{x \in X \mid f(x) = 0\}$ in X is called the kernel of the homomorphism f; it is denoted by $\mathrm{Ker}\, f$. If $f(X) = Y$ (resp., $\mathrm{Ker}\, f = 0$), then f is called a surjective homomorphism (resp., an injective homomorphism or a monomorphism). If f is a surjective monomorphism, then f is called an isomorphism. We denote by $\mathrm{Hom}_{\mathbb{Z}}(X, Y)$ the set of homomorphisms from X into Y; it is an additive group with addition defined by the relation $(f+g)(x) = f(x) + g(x)$ for all $x \in X$.

Let X_1 be a subgroup in the abelian group X. For every $x \in X$, the set $\{x + x_1 \mid x_1 \in X_1\}$ is denoted by $x + X_1$. We denote by X/X_1 the set $\{x + X_1 \mid x \in X\}$. For any two elements $x, y \in X$, we denote by $(x + X_1) + (y + X_1)$ the subset $(x+y) + X_1$ in X/X_1. It may be easily checked that this addition turns X/X_1 into an additive abelian group with zero element $0 + X_1 = X_1$. The group X/X_1 is called the factor group of the group X with respect to the subgroup X_1. The relation $x \to x + X_1$ defines a surjective homomorphism $h: X \to X/X_1$ which is called the natural epimorphism. Every surjective group homomorphism $f: X \to Y$ induces the group isomorphism $\overline{f}: X/\mathrm{Ker}\, f \to Y$ which is defined by the relation $\overline{f}(x + \mathrm{Ker}\, f) = f(x)$.

Let R be a ring, let X be a right (resp., left) R-module, let X_1 be a submodule in X, and let X/X_1 be the factor group of the additive group X with respect to Y. For any two elements $x \in X$ and $a \in R$, we set $(x + X_1)a = xa + X_1$ (resp., $a(x + X_1) = ax + X_1$). It is directly verified that X/X_1 is a right (resp., left) R-module; X/X_1 is called the factor module of X with respect to X_1. For a module M, a submodule of any factor module of M is called a subfactor of M. If R is a ring and I is an ideal of R, then it is directly verified that the factor module R_R/I is turned into a ring, in which multiplication is defined by the relation $(a_1 + I)(a_2 + I) = a_1 a_2 + I$ for all $a_1, a_2 \in R$. The ring R/I is called the factor ring of the ring R with respect to the ideal I.

If X and Y are right (resp., left) R-modules and $f: X \to Y$ is a homomorphism of additive groups such that $f(xa) = f(x)a$ (resp., $(ax)f = a(x)f$) for all $x \in X$ and $a \in R$, then f is called an R-module homomorphism. If $f: X \to Y$ is a module homomorphism and Y' is a submodule of Y, then we denote by $f^{-1}(Y')$ the submodule $\{x \in X \mid f(x) \in Y'\}$ of X. We assume that homomorphisms of right (resp., left) modules act on the elements from the left (resp., from the right). In addition, if $f: X \to Y$ and $g: Y \to Z$

are two homomorphisms of right (resp., left) modules, then the **composition** $gf: X \to Z$ (resp., $fg: X \to Z$ of the homomorphisms f and g is defined by the relation $gf(x) = g(f(x))$ (resp., $(x)fg = ((x)f)g$).

Let A, B be two rings, let X and Y be two A-B-bimodules, and let $f: X \to Y$ be a homomorphism of left A-modules that also is a homomorphism of right B-modules. Then f is called a **homomorphism of A-B-bimodules.** Isomorphisms, endomorphisms and automorphisms of A-B-bimodules are similarly defined. Let X and Y be two rings. A mapping $f: X \to Y$ is called a **ring homomorphism** if f is a homomorphism of additive groups and a homomorphism of multiplicative monoids. A (ring or module) surjective monomorphism is called an **isomorphism.**

Let R be a commutative ring, and let A be a ring. The ring A is called an **algebra** over R or an **R-algebra** if there exists a nonzero ring homomorphism f from R into the center of the ring A. Then $f(R)$ is a central unitary subring in A, and we identify R with $f(R) \subseteq A$ provided that Ker $f = 0$. Thus, if R is a field, then Ker $f = 0$, and we can assume that the ring A is an algebra over the field R if and only if A contains the field R as a central unitary subring.

If X is a module (resp., a ring), then module (ring) homomorphisms $X \to X$ are called module (resp., ring) **endomorphisms** of the module (resp., ring) X, and for any submodule (resp., ideal) X' of X, by the relation $x \to x + X'$ is defined the surjective module (resp., ring) homomorphism $h: X \to X/X'$, called the **natural epimorphism.** The kernel Ker f of any module (resp., ring) homomorphism $f: X \to Y$ is a submodule (resp., an ideal) of the module (resp., of the ring) X, and the relation $\overline{f}(x + \text{Ker } f) = f(x)$ induces the module (ring) isomorphism $\overline{f}: X/\text{Ker } f \to f(X)$. If X is a module (resp., a ring) and Y is a submodule (resp., an ideal) of X, then every submodule (resp., every ideal) of the factor module (resp., of the factor ring) X/Y has the form Y'/Y, where Y' is a submodule (resp., an ideal) of X and $Y \subseteq Y' \subseteq X$, and the relation $\varphi(x + Y') = (x + Y) + Y'/Y$ defines a module (resp., ring) isomorphism $X/Y' \cong (X/Y)/(Y'/Y)$.

For any two right (resp., left) R-modules X and Y, the set of all homomorphisms from X into Y is denoted by $\text{Hom}(X_R, Y_R)$ (resp., $\text{Hom}(_RX, _RY)$); it is a subgroup of the additive group $\text{Hom}_{\mathbb{Z}}(X, Y)$. The additive group $\text{Hom}(X_R, X_R)$ (resp., $\text{Hom}(_RX, _RX)$) is denoted by End X_R (resp., $\text{End}_R\, X$); it is a ring such that the product fg of two endomorphisms f and g coincides with the composition of f and g, i.e. $fg(x) = f(g(x))$ (resp., $(x)fg = ((x)f)g$). The ring End X_R (resp., $\text{End}_R\, X$) is called the **endomorphism ring** of the right module X_R (resp., the left module $_RX$). Every ring R is isomorphic to the endomorphism rings End R_R and $\text{End}_R\, R$; the required ring isomorphisms are the mappings $\varphi\colon R \to \operatorname{End} R_R$ and $\psi\colon R \to \operatorname{End}_R R$ such that $\varphi(a)\colon x \to ax$ and $\psi(a)\colon x \to xa$ for all $x \in R$.

A module (resp., ring) endomorphism is called a module (resp., ring) **automorphism** if it is both injective and surjective. The set of all automorphisms of the module (ring) X is denoted by $\operatorname{Aut}(X)$. It is a group such that for any automorphism f, the inverse automorphism f^{-1} is correctly defined by the relation $f^{-1}(f(x)) = x$ for all $x \in X$. The identity element of the automorphism group is the identity automorphism $1_X\colon x \to x$. The automorphism group of any module is the group of invertible elements of the endomorphism ring of this module. Injective homomorphisms are also called **monomorphisms**, and surjective homomorphisms are called **epimorphisms**.[1]

If X and Y are two right R-modules, then for any $f \in \operatorname{End} Y_R$, $g \in \operatorname{Hom}(X_R, Y_R)$ and $h \in \operatorname{End} X_R$, the composition fgh is contained in the group $\operatorname{Hom}(X_R, Y_R)$. Similarly, if X and Y are left R-modules, then for any $f \in \operatorname{End}_R Y$, $g \in \operatorname{Hom}({}_RX, {}_RY)$ and $h \in \operatorname{End}_R X$, the composition hgf is contained in the group $\operatorname{Hom}({}_RX, {}_RY)$. Therefore, there are natural bimodules ${}_{\operatorname{End} Y_R}\operatorname{Hom}(X_R, Y_R)_{\operatorname{End} X_R}$, and similarly, we have ${}_{\operatorname{End}\,{}_RX}\operatorname{Hom}({}_RX, {}_RY)_{\operatorname{End}\,{}_RY}$.

For a module M, a submodule X of M is said to be **fully invariant** in M if $f(X) \subseteq X$ for any endomorphism f of M. Every fully invariant submodule of the right module M also is a submodule of the left module ${}_{\operatorname{End} M}M$. For a ring R, the ideals of R coincide with the fully invariant submodules in R_R; they also coincide with the fully invariant submodules in ${}_RR$. If a left ideal X of R is not an ideal in R, then X is a submodule of the left $\operatorname{End} R_R$-module R, and X is not a fully invariant submodule of R_R.

Let R be a ring. If X, Y are two subsets of a right (resp., left) R-module M, then we set
$(X \,^{\cdot}_{\cdot}\, Y) = \{a \in R\colon Xa \subseteq Y\}$ and $r_R(X) = r(X) = (X \,^{\cdot}_{\cdot}\, 0) = \{a \in R\colon Xa = 0\}$ (resp., $(Y \,^{\cdot}_{\cdot}\, X) = \{a \in R\colon aX \subseteq Y\}$ and $\ell_R(X) = \ell(X) = (0 \,^{\cdot}_{\cdot}\, X)) = \{a \in R :: aX = 0)$.

The subset $r(X)$ (resp., $\ell(X)$) of the ring R is a right (left) ideal of R; it is called the **right** (resp., **left**) **annihilator** of the subset X of the right (resp., left) module M.

If Y is a submodule of a right (resp., left) module M, then $(X \,^{\cdot}_{\cdot}\, Y)$ (resp., $(Y \,^{\cdot}_{\cdot}\, X)$) is a right (resp., left) ideal of R, and $(X \,^{\cdot}_{\cdot}\, Y)$ (resp., $(Y \,^{\cdot}_{\cdot}\, X)$) is an ideal of R, provided that X and Y are submodules of the right (left) module M.

[1] Here we do not consider epimorphisms and monomorphisms in the categorical sense. We only note that the category of rings has nonsurjective epimorphisms, but natural ring epimorphisms onto factor rings are surjective, and all epimorphisms (resp., monomorphisms) in the category of right R-modules are surjective (resp., injective).

It is clear that $r(X) = \cap_{x\in X} r(x)$ (resp., $\ell(X) = \cap_{x\in X}\ell(x)$) and $r(X)$ (resp., $\ell(X)$) is an ideal of R if X is a submodule of the right (resp., left) module M.

A module with zero annihilator is called a faithful module. For a ring R, every right R-module M can be naturally turned into a faithful right module over the ring $R/r(M)$. In addition, if we associate with any element $m \in M$ the homomorphism $f_m \in \mathrm{Hom}(R_R, M)$ such that $f_m(a) = ma$ for all $a \in R$, then we have an End M-R-bimodule isomorphism $M \to \mathrm{Hom}(R_R, M)$.

For any cyclic module xR, the mapping $f : a \to xa$ is a module epimorphism from R_R onto xR with kernel $r(x)$; it induces the module isomorphism $R_R/r(x) \cong xR$. Similarly, $Rx \cong_R R/\ell(x)$. This implies that all cyclic right (resp., left) R-modules coincide, up to isomorphism, with the factor modules R/B with respect to right (resp., left) ideals B and cardinalities of all cyclic R-modules are upper bounded by the cardinality of R. Therefore, there exists a set $\mathcal{E}$ of cyclic right R-modules such that any cyclic right R-module is isomorphic to some module in $\mathcal{E}$.[2]

If R is a ring and X is a subset in R, then $r(\ell(r(X))) = r(X)$ and $\ell(r(\ell(X))) = \ell(X)$. If Q is a ring and R is a subring in Q, then for every subset X in R, we have that $r_R(X) = R \cap r_Q(X)$ and $\ell_R(X) = R \cap \ell_Q(X)$.

Theorem 1.1 *For a ring R, the following conditions are equivalent.*

1. *R is a ring with maximum condition on right annihilators.*
2. *R is a ring with minimum condition on left annihilators.*
3. *R is a subring of a ring with maximum condition on right annihilators.*
4. *R is a subring of a ring with minimum condition on left annihilators.*

In this case, any subset B in R contains a finite subset $B^* = \{b_1, \ldots, b_n\}$ such that

$$r(B) = r(B^*) = r(b_1) \cap \cdots \cap r(b_n).$$

Theorem 1.2 *For a ring R, the following conditions are equivalent.*

1. *For every subset $X \subseteq R$, there exists an idempotent $e \in R$ with $r(X) = eR$.*
2. *For every subset $Y \subseteq R$, there exists an idempotent $f \in R$ with $\ell(Y) = Rf$.*

[2] The similar assertion does not hold for all right R-modules, since there does not exist a common upper bound of cardinalities of all right R-modules.

Proof It is sufficient to prove the implication (1) $\Rightarrow$ (2). We set $X = \ell(Y)$. We have $X = \ell(r(X))$. By assumption, $r(X) = eR$ for some idempotent $e \in R$. Then $Y = \ell(eR) = R(1-e)$, and we set $f = 1 - e$. □

A ring R is called a Baer ring if it satisfies the preceding equivalent conditions (1) and (2).

1.2 Simple and Semisimple Modules

A nonzero module M is said to be a simple module if it has no nonzero proper submodule – that is, the only submodules of M are zero module and the module M itself. A ring R is said to be a simple ring if R has no nonzero proper two-sided ideal.

Let M a nonzero module. A submodule N of M is said to be a maximal submodule if N is a maximal element of the nonempty set of all proper submodules of M with respect to set-inclusion. The set of all maximal submodules of M is denoted by $\max M$.

For a module M, the intersection of kernels of all homomorphisms from M into arbitrary simple modules is called the Jacobson radical of M; it is denoted by $J(M)$. It is clear that either $J(M) = M$ (if $\max M = \varnothing$) or $J(M)$ coincides with the intersection of all maximal submodules of M (if $\max M \neq \varnothing$). A module M is said to be a semiprimitive module if $J(M) = 0$.

For a module M, a submodule X of M is said to be small (in M) if $X + Y \neq M$ for any proper submodule Y of M. If X is a small submodule of M, then we express it as $X \subset_s M$. Every nonzero submodule has a simple subfactor.

Theorem 1.3 *For a nonzero module M, the following conditions are equivalent.*

1. *M is the direct sum of simple modules.*
2. *M is the sum of simple modules.*
3. *Every submodule of M is a direct summand of M.*

A module M is said to be semisimple if M satisfies the preceding equivalent conditions.

Theorem 1.4 (Wedderburn–Artin theorem) *For a ring R, the following conditions are equivalent.*

1. *R is a right (resp., left) semisimple ring.*
2. *Each right (resp., left) R-module is semisimple.*
3. *R is a finite direct product of simple Artinian rings.*
4. *R is isomorphic to a finite direct product of matrix rings over division rings.*

For a module M, the sum of all simple submodules of M is called the socle of M; it is denoted by $\mathrm{Soc}(M)$. If M has no simple submodules, then $\mathrm{Soc}(M) = 0$ by definition. Clearly, $\mathrm{Soc}(M)$ is a fully invariant submodule of M, and it is the largest semisimple submodule of M.

Definition 1.5 A right R-module M is called a semi-Artinian module if for every submodule $N \neq M$, $\mathrm{Soc}(M/N) \neq 0$. A ring R is called a right semi-Artinian ring if R_R is semi-Artinian.

If X is a module (resp., a ring) and $\{Y_i\}_{i\in I}$ is a set of submodules (resp., ideals) of X with $\cap_{i\in I} Y_i = 0$, then X is called a subdirect product of the factor modules X/Y_i (resp., factor rings X/Y_i). In this case, if at least one of the modules (resp., ideals) Y_i is equal to zero, then the subdirect product is said to be trivial. If $Y_i \neq 0$ for all i, then the subdirect product is said to be nontrivial. A nonzero module (resp., ring) X is said to be subdirectly indecomposable if X is not a nontrivial subdirect product of any factor modules (resp., factor rings) of X; i.e. the intersection of all nonzero submodules (resp., ideals) of X does not equal to zero. This means that there exists a nonzero submodule (resp., ideal) of X contained in every nonzero submodule (resp., ideal) of X. Every nonzero module (resp., ring) is a subdirect product of subdirectly indecomposable modules (resp., rings).

For a module M, the Jacobson radical $J(M)$ is fully invariant in M, $M/J(M)$ is a semiprimitive module and $M/J(M)$ is a subdirect product of simple modules. The Jacobson radical $J(M)$ contains each small submodule X of M and every finitely generated submodule $N \subseteq J(M)$ is small in M. Therefore, $J(M)$ is the sum of all small submodules of M. If $J(M)$ is small in M, then $J(M)$ is the largest small submodule of M. In addition, if M is a nonzero finitely generated module, then $M \neq J(M)$ and $J(M)$ is the largest small submodule of M.

For any ring R, we have $J(R_R) = J({}_RR)$; this ideal is denoted by $J(R)$. The ideal $J(R)$ is called the Jacobson radical of the ring R. It coincides with the largest ideal I in R with the property that $1 - x$ is invertible in R for all $x \in I$.

1.3 Essential and Closed Submodules

Let M be a module. If X is a submodule of M such that $X \cap Y \neq 0$ for any nonzero submodule Y of M, then X is said to be essential in M. In this case, we say that M is an essential extension of X, and we express it as $X \subset_e M$. A submodule Y of M is said to be closed (in M) if Y coincides with any submodule in M that is an essential extension of Y. If $X, \overline{X}$ are submodules of M and $\overline{X}$ is a closed (in M) essential extension of X, then $\overline{X}$ is called the closure of X in M.

Let X be a submodule of M, and let Y be a closed submodule of M such that $X \cap Y = 0$, M is an essential extension of $X \oplus Y$, and $X \cap Y' \neq 0$ for any submodule Y' of M properly containing Y. Then Y is called a $\cap$-complement to X in M. A submodule Y of M is said to be $\cap$-complement if Y is a $\cap$-complement in M to some submodule of M.

We list some useful facts in the next theorem.

Theorem 1.6 *Let M be a nonzero module over a ring R.*

1. *If M is an essential extension of a module X, then M is an essential extension of any essential submodule Y of X, X contains the socle of M, and N is an essential extension of $X \cap N$ for any submodule N of M.*
2. *Every direct summand of M is closed in M, and every closed submodule X of M coincides with the closure of X.*
3. *If M is an essential extension of some module N, then for every module homomorphism $f : X \to M$, the submodule $f^{-1}(N)$ is essential in X, where $f^{-1}(N) = \{x \in X : f(x) \in N\}$.*
4. *If $M = \oplus_{i \in I} M_i$ and each M_i is an essential extension of some module X_i, $i \in I$ then M is an essential extension of $\oplus_{i \in I} X_i$.*
5. *M is an essential extension of the direct sum of its nonzero cyclic submodules.*
6. *Every submodule X of M has at least one closure $\overline{X}$, and $\overline{X} \cap Y = 0$ for any submodule Y of M with $X \cap Y = 0$.*
7. *If the sum $\sum_{i \in I} X_i$ of submodules X_i of M is a direct sum, then the sum of closures $\sum_{i \in I} \overline{X}_i$ of submodules X_i in M is a direct sum.*
8. *If X is a submodule of M, then for any submodule Y of M with $X \cap Y = 0$ (e.g. for $Y = 0$), the module M contains at least one closed $\cap$-complement $\overline{Y}$ to X with $\overline{Y} \supseteq Y$. Therefore, X has at least one closed $\cap$-complement Y and at least one closure $\overline{X}$ such $\overline{X} \cap Y = 0$, and X is a direct summand of the essential submodule $X' = X \oplus Y$ in M. In particular, X is a direct summand of some essential submodule of M.*

9. *Let X be a submodule of M, let Z be a closure of X in M, and let Y be a $\cap$-complement to X in M. Then Z is a $\cap$-complement to Y in M, and Y is closed in M.*
10. *The set of all closed submodules of M coincides with the set of all $\cap$-complement submodules of M.*
11. *For any module N and each homomorphism $f : X \to N$, there exists an essential submodule X' of M and a homomorphism $f' : X' \to N$ such that f' coincides with f on X, $f'(X') = f(X)$, $X' = X \oplus Y$, Y is a closed $\cap$-complement to X in M and $f'(Y) = 0$.*

Definition 1.7 A nonzero module M is called a uniform module if any two nonzero (cyclic) submodules of M intersect nontrivially.

It is clear that every subdirectly indecomposable module is uniform.

Theorem 1.8 *For the module M, the following conditions are equivalent.*

1. *M is a uniform module.*
2. *M is an essential extension of a uniform module.*
3. *M is an essential extension of any nonzero (cyclic) submodule of M.*
4. *Any two closed nonzero submodules of M have the nonzero intersection.*

Definition 1.9 A module M is said to be finite-dimensional if M does not contain an infinite direct sum of nonzero submodules.

All uniform modules are finite-dimensional.

Theorem 1.10 *For the module M, the following conditions are equivalent.*

1. *M is a finite-dimensional module.*
2. *M is an essential extension of a finite direct sum of finite-dimensional modules.*
3. *M does not contain an infinite direct sum of closed submodules.*
4. *M is a module with maximum condition on closed submodules.*
5. *M is a module with minimum condition on closed submodules.*
6. *Any submodule of M is an essential extension of a finitely generated module.*
7. *There exists a finite set $\{X_i\}_{i=1}^k$ of submodules X_i of M such that $\cap_{i=1}^k X_i = 0$ and M/X_i is a uniform module, $i = 1, \ldots, k$.*
8. *There exists a positive integer n such that M does not contain a direct sum of $n + 1$ nonzero modules, and M is an essential extension of the direct sum of n nonzero uniform modules.*

The integer n from condition (6) is called the uniform dimension or the Goldie dimension of the finite-dimensional module M; it is denoted