19 Post-Main-Sequence Evolution: Low-Mass Stars

As a star ages, more and more of the hydrogen in its core becomes consumed by fusion into helium. Once this core hydrogen is used up, how does the star react and adjust? Without the hydrogen fusion (H-fusion) to supply its luminosity, one might think that perhaps the star would simply shrink, cool, and dim, and so die out, much like a candle when all its wax is used up.

Instead it turns out that stars at this post-main-sequence stage of life actually start to expand at first keeping roughly the same luminosity and so becoming cooler at the surface, but eventually becoming much brighter giant or supergiant stars, shining with a luminosity that can be thousands times that of their core-hydrogen-burning main sequence.

However, as illustrated in Figure 19.1, the detailed evolution and final states of stars depends on the stellar mass, with distinct difference for stars with initial masses below versus above about $8 M_\odot$. Figure 19.2 summarizes the corresponding post-main-sequence evolutionary tracks in the H–R diagram for the Sun (top) and for stars with mass up to $10\, M_\odot$ (bottom).

The remainder of this chapter focuses our initial attention on solar-type stars with $M \lesssim 8 M_\odot$. The evolution and final states of high-mass stars are discussed in Chapter 20.

19.1 Hydrogen-Shell Burning and Evolution to the Red-Giant Branch

The apparently counterintuitive *post*-main-sequence adjustment of stars can actually be understood through the same basic principles used to understand their initial, *pre*-main-sequence evolution. When the core runs out of hydrogen fuel, the lack of energy generation does indeed cause the core itself to contract. But the result is to make this core even denser and hotter. Then, much as the hot coals at the heart of a wood fire help burn the wood fuel around it much faster, the higher temperature of a contracted stellar core actually makes the overlying *shell* of hydrogen fuel around the core burn even more vigorously!

Such shell burning of hydrogen actually makes the star expand. Unlike during the main sequence – when there is an essential regulation or compatibility between the luminosity generated in the core and the luminosity that the radiative envelope is able to transport to the stellar surface – this shell-burning core is actually overluminous

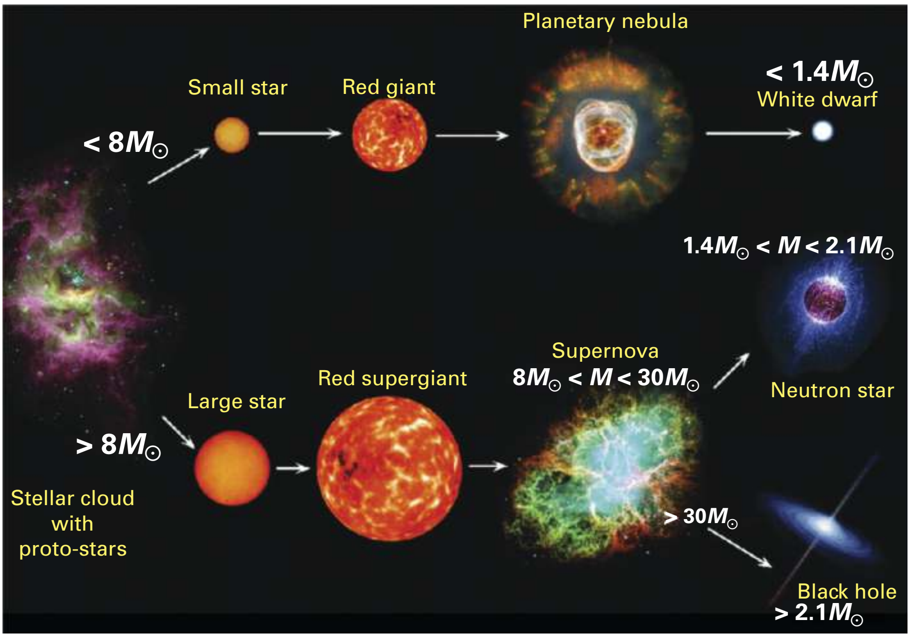

Figure 19.1 Distinct evolution and final states for stars with initial masses above and below $8M_\odot$.

relative to the envelope luminosity that is set by the $L \sim M^3$ scaling law. As such, instead of emitting this core luminosity as surface radiation, the excess energy acts to expand the star, in effect doing work against gravity to reverse the Kelvin–Helmholtz contraction that occurred during the star's pre-main-sequence evolution. Initially, the radiative envelope keeps the luminosity fixed so that, as the star expands, the surface temperature again declines, with the star thus again evolving horizontally on the H–R diagram, this time from left to right.

But as the surface temperature approaches the limiting value $T \approx 3500$–4000 K, the envelope again becomes more and more convective, which thus now allows this full high luminosity of the hydrogen-shell-burning core to be transported to the surface. The star's luminosity thus increases, now with the temperature staying nearly constant at the cool value for the Hayashi limit. In the H–R diagram, the star essentially climbs back up the Hayashi track, eventually reaching the region of the cool, red giants in the upper right of the H–R diagram.

The above describes a general process for all stars, but the specifics depend on the stellar mass. For masses less than the Sun, the main-sequence temperature is already quite close to the cool limit, so evolution can proceed almost directly vertically up the Hayashi track. For masses much greater than the Sun, the luminosity and temperature on the main sequence are both much higher, and so the horizontal evolutionary phase is more sustained. And since the luminosity is already very high, these stars become red supergiants without ever having to reach or climb the Hayashi track.

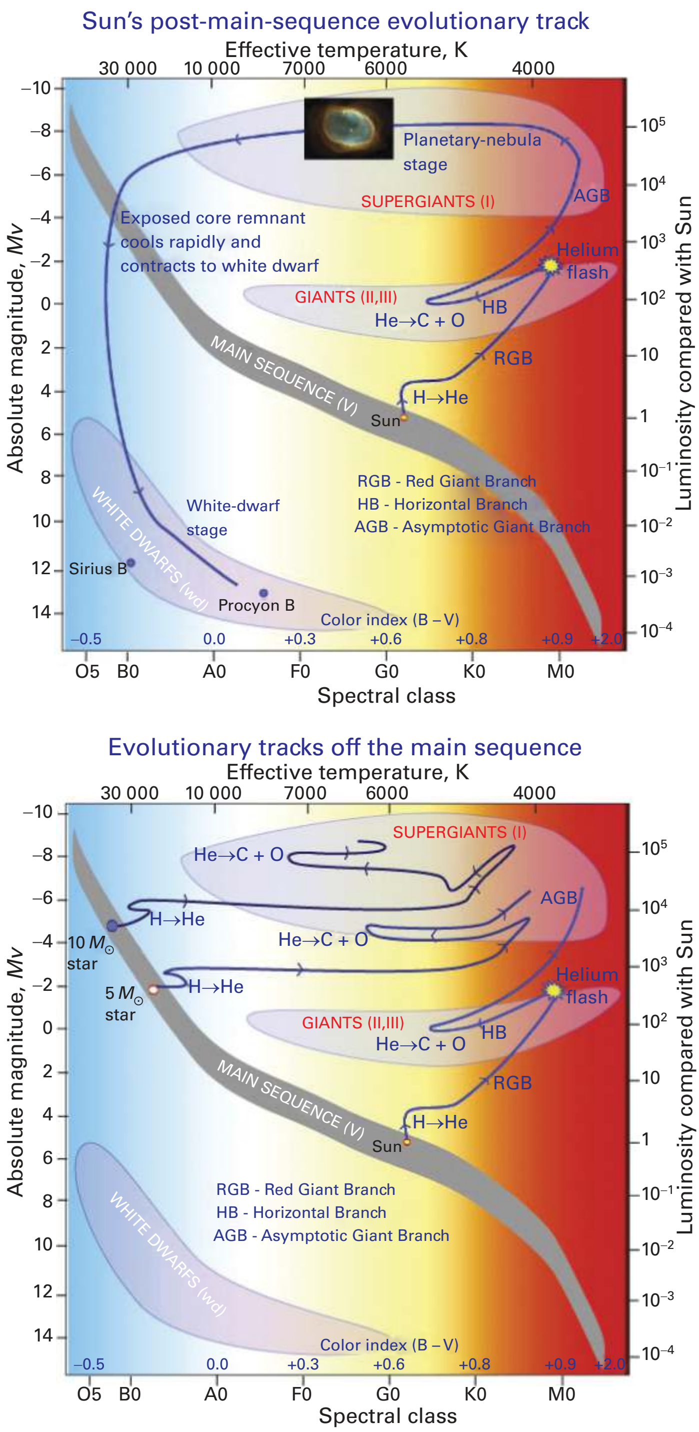

Figure 19.2 Schematic H–R diagrams to show the post-main-sequence evolution for a solar-mass star (top), and for stars with $M = 1$, 5, and $10 M_\odot$ (bottom). Credit: CSIRO Radio Astronomy Image Archive.

19.2 Helium Flash to Horizontal Branch Core Burning

This hydrogen-shell burning also has the effect of increasing further the temperature of the stellar core, and eventually this reaches a level where the fusion of the helium itself becomes possible, through what is known as the "triple-α process,"[1]

$$3\,{}_4\mathrm{He}^{+2} \leftrightharpoons {}_8\mathrm{Be}^{+4} + {}_4\,\mathrm{He}^{+2} \rightarrow {}_{12}\mathrm{C}^{+6}. \tag{19.1}$$

The direct fusion of two ${}_4\mathrm{He}^{+2}$ nuclei initially makes an unstable nucleus of berylium (${}_8\mathrm{Be}^{+4}$), which usually just decays back into helium. But if the density and temperature are sufficiently high, then during the brief lifetime of the unstable berylium nucleus, another helium can fuse with it to make a very stable carbon nucleus ${}_{12}\mathrm{C}^{+6}$. Since the final step of fusing ${}_4\mathrm{He}^{+2}$ and ${}_8\mathrm{Be}^{+4}$ involves overcoming an electrostatic repulsion that is $2 \times 4 = 8$ times higher than for proton–proton (PP) fusion of hydrogen, helium fusion (He-fusion) requires a much higher core temperature, $T_{\mathrm{He}} \approx 8 \times 15\,\mathrm{MK} \approx 120\,\mathrm{MK}$.

In stars with more than a few solar masses, this ignition of the helium in the core occurs gradually, since the higher core temperature from the addition of helium burning (He-burning) increases the gas pressure, making the core tend to expand in a way that regulates the burning rate.

In contrast, for the Sun and other stars with masses $M < 2M_{\odot}$, the number density of electrons n_{e} in the helium core is so high[2] that their core becomes *electron degenerate*. As discussed in Section 18.3 for the brown-dwarf stars that define the lower mass limit for hydrogen burning (H-burning), electron degeneracy occurs when the mean distance between electrons $\sim n_{\mathrm{e}}^{-1/3}$ becomes comparable to the de Broglie wavelength $\bar{\lambda}_{\mathrm{e}} = \bar{h}/p_{\mathrm{e}}$. The properties of such degeneracy are discussed further in Section 19.4 on the degenerate white-dwarf end-states of solar-type stars.

But in the present context, a key point is that the response to any heat addition is quite different than for an ideal gas. By the Virial Theorem for a gravitationally bound ideal gas, the added heating from any increase in nuclear burning leads to an expansion that cools the gas, thus reducing the burning and so keeping it stable. In contrast, for a degenerate gas, the expansion from adding heat actually makes the temperature increase even further. Thus, once the evolutionary increase in core temperature reaches a level that ignites fusion of helium into carbon, the degenerate nature of the gas leads to a *helium flash*, in which a substantial fraction of the core of helium is fused into carbon over a very short timescale.

As illustrated in the top panel of Figure 19.2, this flash marks the "tip" of the Red Giant Branch (RGB) in the H–R diagram. Somewhat surprisingly, the sudden

[1] Because helium nuclei are sometimes referred as "α-particles." In this formula, the left subscript denotes the atomic mass (number of proton and neutrons), while the right superscript denotes the nuclear charge (number of protons).

[2] Recall that, on the main sequence, the radii of stars is (very) roughly proportional to their mass, $R \sim M$. But since density scales as $\rho \sim M/R^3$, the *density* of low-mass stars tends generally to be higher than in high-mass stars, roughly scaling as $\rho \sim 1/M^2$. This overall scaling of average stellar density also applies to the relative densities of stellar cores, and so helps to explain why the cores of low-mass stars tend to become electron degenerate, while those of higher-mass stars do not.

addition of energy is largely absorbed by the expansion of the core and the overlying stellar envelope. Since the expanded core is no longer very degenerate, the star thus simply settles down to a more quiescent, stable phase of He-burning. The expanded core also means the shell burning of hydrogen actually declines, causing the luminosity to decrease from the tip of the Red Giant Branch, where the helium flash occurs, to a somewhat hotter, dimmer region known as the Horizontal Branch in the H–R diagram.

This Horizontal Branch (HB) can be loosely thought of as the He-burning analog of the H-burning main sequence, but a key difference is that it lasts a much shorter time, typically only 10 to 100 *million years*, much less than the many billion years for a solar mass star on the main sequence. This is partly because the luminosity for HB stars is so much higher than for a similar mass on the main sequence, implying a much higher burn rate of fuel. But another factor is that the energy yield per-unit-mass, ϵ, for He-fusion to carbon is about a tenth of that for H-fusion to helium, namely about $\epsilon_{\rm He} \approx 0.06$ percent versus the $\epsilon_{\rm H} \approx 0.7$ percent for H-burning (see Figure 20.1). With the lower energy produced, and the higher rate of energy lost in luminosity, the lifetime is accordingly shorter.

19.3 Asymptotic Giant Branch to Planetary Nebula to White Dwarf

Once the core runs out of helium, He-burning also shifts to an inner shell around the core, which itself is still surrounded by a outer shell of more vigorous H-burning. This again tends to increase the core luminosity, but now, because the star is cool and thus mostly convective, this energy is mostly transported to the surface with only a modest further expansion of the stellar radius. This causes the star to again climb in luminosity along what is called the Asymptotic Giant Branch (AGB), which parallels the Hayashi track at just a somewhat hotter surface temperature.

In the Sun and stars of somewhat higher mass, up to $M \lesssim 8 M_\odot$, there can be further ignition of the carbon fusing with helium to form oxygen. But further synthesis up the periodic table requires overcoming the greater electrical repulsion of more highly charged nuclei. This, in turn, requires a temperature higher than occurs in the cores of lower-mass stars, for which the onset of electron degeneracy prevents contraction to a denser, hotter core. Further core burning thus ceases, leaving the core as an inert, degenerate ball of carbon and oxygen, with final mass on order of $1\,M_\odot$, with the remaining mass contained in the surrounding envelope of mostly hydrogen.

But such AGB stars tend also to be pulsationally unstable and, because of the very low surface gravity, such pulsations can over time actually eject the entire stellar envelope. This forms a circumstellar *nebula* that is heated and ionized by the very hot remnant core. As seen in the left panel of Figure 20.4, the resulting circular nebular emission glow somewhat resembles the visible disk of a planet, so these are called “planetary nebulae,” though they really have nothing much to do with actual planets. After a few thousand years, the planetary nebula dissipates, leaving behind just the degenerate remnant core, a white-dwarf star.

19.4 White-Dwarf Stars

The electron-degenerate nature of white-dwarf stars endows them with some rather peculiar, even extreme, properties. As noted, they typically consist of roughly a solar mass of carbon and oxygen, but have a radius comparable to that of the Earth, $R_e \approx R_\odot/100$. This small radius makes them very dense, with $\rho_{wd} \approx 10^6 \bar{\rho}_\odot \approx 10^6$ g/cm^3, i.e., about a million times (!) the density of water, and so a million times the density of normal main-sequence stars such as the Sun. It also gives them very strong surface gravity, with $g_{wd} \approx 10^4 g_\odot \approx 10^6$ m/s^2, or about 100 000 times Earth's gravity!

As noted in Section 18.3 for the brown-dwarf stars that define the lower mass limit for H-burning, a gas becomes electron degenerate when the electron number density n_e becomes so high that the mean distance between electrons becomes comparable to their reduced de Broglie wavelength,

$$n_e^{-1/3} \approx \bar{\lambda} \equiv \frac{\hbar}{p_e} = \frac{\hbar}{m_e v_e}, \tag{19.2}$$

where the electron thermal momentum p_e equals the product of its mass m_e and thermal speed v_e, and $\hbar \equiv h/2\pi$ is the reduced Planck constant. The associated electron pressure is

$$P_e = n_e v_e p_e = n_e^{5/3} \frac{\hbar^2}{m_e} = \left(\frac{\rho Z}{A m_p}\right)^{5/3} \frac{\hbar^2}{m_e}, \tag{19.3}$$

where the last equality uses the relation between electron density and mass density, $\rho = n_e A m_p / Z$, with Z and $A m_p$ the average nuclear charge and atomic mass. For example, for a carbon white dwarf, the atomic number $Z = 6$ gives the number of free (ionized) electrons needed to balance the $+Z$ charge of the carbon nucleus, while the atomic weight $A m_p = 12 m_p$ gives the associated mass from the carbon atoms. The hydrostatic equilibrium (cf. Eq. (15.1)) for pressure-gradient support against gravity then requires, for a white-dwarf star with mass M_{wd} and radius R_{wd},

$$\frac{P_e}{R_{wd}} \approx \rho \frac{G M_{wd}}{R_{wd}^2}. \tag{19.4}$$

Using the density scaling $\rho \sim M_{wd}/R_{wd}^3$, we can combine Eqs. (19.3) and (19.4) to solve for a relation between the white-dwarf radius and its mass,

$$R_{wd} \approx \frac{3.6}{G M_{wd}^{1/3}} \frac{\hbar^2}{m_e} \left(\frac{Z}{A m_p}\right)^{5/3} \approx \boxed{0.01 R_\odot \left(\frac{M_\odot}{M_{wd}}\right)^{1/3}}, \tag{19.5}$$

where the approximate evaluation uses the fact that for both carbon and oxygen the ratio $Z/A = 1/2$, and the factor 3.6 comes from a detailed computation not covered here. For a typical mass of order the solar mass, we see that a white dwarf is very compact, comparable to the radius of the Earth, $R_e \approx 0.01 R_\odot$. But note that this radius actually *decreases* with increasing mass.

19.5 Chandrasekhar Limit for White-Dwarf Mass: $M < 1.4M_\odot$

This fact that white-dwarf radii decrease with higher mass means that, to provide the higher pressure to support the stronger gravity, the electron speed v_e must strongly increase with mass. Indeed, at some point this speed approaches the speed of light, $v_e \approx c$, implying that the associated electron pressure now takes the scaling (cf. Eq. (19.3)),

$$P_e = n_e c p_e = n_e^{4/3} \hbar c = \left(\frac{\rho Z}{A m_p} \right)^{4/3} \hbar c. \tag{19.6}$$

Applying this in the hydrostatic relation (19.4), we now find that the radius R cancels! Instead we can solve for an upper limit for a white-dwarf's mass,

$$M_{wd} \le M_{ch} = \sqrt{3\pi} \left(\frac{\hbar c}{G} \right)^{3/2} \left(\frac{Z}{A m_p} \right)^2 \approx \boxed{1.4 M_\odot,} \tag{19.7}$$

where the subscript refers to "Chandrasekhar," the astrophysicist who first derived this mass limit, and the proportionality factor $\sqrt{3\pi}$ comes from a detailed calculation beyond the scope of the discussion here.

As discussed in later sections (see, for example, Section 31.1), when accretion of matter from a binary companion puts a white dwarf over this limit, it triggers an enormous "white-dwarf supernova" explosion, with a large, relatively well-defined peak luminosity, $L \approx 10^{10} L_\odot$. This provides a very bright standard candle that can be used to determine distances as far as 1 Gpc, giving a key way to calibrate the expansion rate of the universe.

But in the present context, this limit means that sufficiently massive stars with cores above this mass cannot end their lives as a white dwarf. Instead, they end as violent "core-collapse supernovae," leaving behind an even more-compact final remnant, either a neutron star or black hole, as we discuss in the next chapter.

19.6 Questions and Exercises

Quick Questions

1. Note that pressure can be written as $P = Nvp$ for speed v and momentum $p = mv$ in a medium with N atoms of mass m. Show that this recovers the ideal gas form for pressure by identifying the thermal speed scaling with temperature, $kT/2 = mv_{th}^2/2$.
2. Derive a general formula for the escape speed v_{esc} for a white dwarf of mass M. What is the value (in km/s) for $M = M_\odot$ and how does this compare with the Sun's escape speed?
3. What is the maximum escape speed (in km/s) of a white dwarf?
4. Derive a general formula for $\log g$ of a white-dwarf star of mass M, and compare this with $\log g_\odot$.

Exercises

1. *Planetary nebula emission and expansion*
   Suppose we observe the Hα line from a spherical planetary nebula, and find it has a maximum wavelength of 656.32 nm and minimum wavelength of 656.24.
   a. Estimate how fast the nebula is expanding. Give your answer in both km/s and au/year.
   b. Now suppose that over a time of 10 years the nebula's angular diameter is observed to expand from 10 arcsec to 10.1 arcsec. What is the distance to the nebula (in pc)?
   c. What is actual physical diameter of the nebula (in au)?
2. *White-dwarf cooling time*
   a. How many carbon nuclei are there in a pure-carbon white dwarf of mass 1 $M_\odot$?
   b. If the carbon is full ionized, how many electrons are there?
   c. Each particle in a gas (even a degenerate gas) of temperature T has a thermal energy $(3/2)kT$. If this fully ionized carbon white dwarf has a constant temperature of 10^7 K through nearly all of its interior mass, what is its total thermal energy (in J)?
   d. Assuming it has a radius of 0.01 $R_\odot$ and radiates like a 30 000 K blackbody, compute its luminosity, $L_{\rm wd}$ (in $L_\odot$).
   e. Now use these results to estimate this white dwarf's cooling lifetime, $t_{\rm cool}$ (in yr).
3. *White-dwarf supernova*
   Consider a pure-carbon white dwarf at the Chandrasekhar mass limit that collapses suddenly from a radius $R \approx R_{\rm e}$, igniting a sudden conversion of the carbon to iron.
   a. Considering the relative binding energy per nucleon of iron versus carbon (e.g., from Figure 20.1), estimate the associated nuclear energy release $E_{\rm n}$ (in J) if the entire stellar mass is converted to iron.
   b. Assuming the initial white dwarf has a uniform density, compute its gravitational binding energy $E_{\rm g}$, and compare this with the nuclear energy release $E_{\rm n}$ from part a.
   c. Assuming just 1 percent of this nuclear energy is radiated as light within two weeks of the collapse, estimate the average luminosity $L_{\rm avg}$, in units of $L_\odot$.
   d. If the *peak* luminosity during this time is twice this average (i.e. $L_{\rm peak} = 2L_{\rm avg}$), what is the absolute magnitude, $M_{\rm peak}$, at peak brightness?
   e. Using a telescope with a detection limit of apparent magnitude $m = +20$, what is maximum distance d (in Mpc) that the peak brightness of such an explosion can be detected?
4. *Helium stars 4*
   This exercise builds further on the helium-star exercises in Chapters 15, 17, and 18.
   a. For a fully ionized mixture in which the hydrogen, helium, and metal mass fractions are now $X = 0$, $Y = 0.98$, and $Z = 0.02$, compute again the mean molecular weight $\bar{\mu}$ for such a helium star.

b. Use this to generalize Eq. (15.9) to give the scaling of the interior temperature T_{int} of a helium star as a function of its mass M and radius R.
c. As discussed following Eq. (19.1), assume that fusion of He $\rightarrow$ C requires a temperature that is a factor $4 \times 2 = 8$ higher than the $\sim 15\,\text{MK}$ for H-fusion. Now set this $T_{\text{He-fusion}} = T_{int}$ to derive a radius–mass scaling $R(M)$ for such stars on the "helium main sequence."
d. Again use the scaling in Chapter 17 to derive helium-modified mass–luminosity scaling, $L(M)$.
e. Combine the results for c and d to estimate for such helium-main-sequence values for the constants A and b in the scaling relation $L_{\text{HeMS}}/L_{\odot} \approx A(T/T_{\odot})^b$.

5. *Algol paradox and binary mass exchange*
The system Algol includes a short period ($P = 2.87$ days) binary consisting of an early-type main-sequence star (Aa1, B8V, $M = 3.2M_{\odot}$) and a later-type subgiant (Aa2, K0IV, $M = 0.7M_{\odot}$). The orbit is nearly circular.
a. Explain why this represents a "paradox" in the context of single-star evolution.
b. What is the binary separation $a = a_1 + a_2$ (in au)?
c. What are the distances of each component a_1, a_2 from the center of mass?
d. Relative to this center of mass, what is the location of the Roche point between the stars? (See Exercise 2 of Chapter 7.)
e. Suppose the mass ratio between the stars was once reversed; what could happen when the initially more-massive star evolved from its main sequence, to a radius that reaches this Roche point?
f. Discuss how this can resolve the Algol paradox.