Numerical Relativity: Starting from Scratch

Numerical relativity has emerged as the key tool to model gravitational waves – recently detected for the first time – emitted when black holes or neutron stars collide. This book provides a pedagogical, accessible, and concise introduction to the subject. Relying heavily on analogies with Newtonian gravity, scalar fields, and electromagnetic fields, it introduces key concepts of numerical relativity in contexts familiar to readers without prior expertise in general relativity. Readers can explore these concepts by working through numerous exercises and can see them “in action” by experimenting with the accompanying Python sample codes and so develop familiarity with many techniques commonly employed by publicly available numerical relativity codes. This is an attractive, student-friendly resource for short courses on numerical relativity and it also provides supplementary reading for courses on general relativity and computational physics.

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Numerical Relativity: Starting from Scratch

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Preface

General relativity, Einstein’s theory of relativistic gravitation, provides the foundation upon which black holes and neutron stars, compact binary mergers, gravitational waves, cosmology, and all other strong-field gravitational phenomena are constructed. Solutions to Einstein’s equations, except for a few special cases characterized by high degrees of symmetry, cannot be achieved by analytical means alone for many important dynamical scenarios thought to occur in nature. With the aid of computers, however, it is possible to tackle these highly nonlinear equations numerically in order to examine these scenarios in detail. That is the main objective of numerical relativity, the art and science of casting Einstein’s equations in a form suitable for applying general relativity to physically realistic, high-velocity, strong-field dynamical systems and then designing numerical algorithms to solve them on a computer.

Until a little over a dozen years ago, there did not exist a single textbook focusing on numerical relativity. Since then, however, several volumes have appeared, including those by Bona and Palenzuela (2005); Alcubierre (2008); Bona et al. (2009); Gourgoulhon (2012) and Shibata (2016), as well as our own textbook, Baumgarte and Shapiro (2010). One might wonder whether there is any need to add yet another book to this list. However, we believe that this short volume complements the above references rather than duplicating any one of them, as we will explain below.

Several events and developments have occurred since the appearance of most of the above treatises. Most significantly, gravitational waves have been detected directly for the first time, confirming a major prediction of Einstein’s theory and launching the field of gravitational wave astronomy. In 2015, a gravitational wave signal originating from
a binary black-hole system that coalesced in a distant galaxy about a billion years ago was observed on Earth by the Laser Interferometer Gravitational Wave Observatory (LIGO). This detection represented a true milestone in gravitational physics, which was acknowledged by the awarding of a Nobel Prize in 2017 to Barry Barish, Kip Thorne, and Rainer Weiss, the physicists who led the construction of LIGO and the search for gravitational waves. Since then, additional gravitational wave signals from binary black-hole mergers have been observed and analyzed. In 2017 the first gravitational wave signal from the merger of a binary neutron star was detected and it was accompanied by the observation of a gamma-ray burst, together with other electromagnetic radiation. A gravitational wave detection simultaneous with the observation of counterpart electromagnetic radiation represents the holy grail of “multimessenger astronomy”, a golden moment in this growing field. We will discuss some of these observations in more detail in Sections 1.3.2 and 5.1.

These ground-breaking discoveries were met with profound interest and fascination by scientists and the general public alike. They also helped to attract a broader community to the field of numerical relativity, which has emerged as the major tool needed to predict and interpret gravitational wave signals from cosmic events, such as compact-binary mergers. In the past, numerical relativity was practiced predominantly by scientists with appreciable expertise in both general relativity and computational physics, but these new discoveries have prompted broader segments of the physics and astrophysics communities to make use of this tool.

In an independent development, rather sophisticated and mature numerical relativity codes have become publicly and freely available, thereby enabling broader access to the tools of numerical relativity. While, in the past, most numerical relativity groups developed and utilized their own codes, many individuals both within and outside traditional numerical relativity circles are beginning to rely on these “open-source” community codes for their investigations.

This broadening of interest in numerical relativity coupled with the greater availability of numerical relativity codes has spurred a growth in the audience for numerical relativity textbooks. Traditionally, the readers of such books, already well-trained in basic general relativity, are interested in a comprehensive and thorough development of the subject, including a complete treatment of the mathematical underpinnings and full derivation of all the equations. The textbooks listed
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above are aimed at that readership. However, many researchers newly attracted to numerical relativity, including current and future users of community codes, may not have that background and may find the currently available books on numerical relativity inaccessible. We suspect, nevertheless, that many such users may want to gain at least an intuitive understanding of some of the variables, key equations, and basic algorithms often encountered in numerical relativity, rather than using community codes solely as “black boxes”. This book is aimed at those readers.

The purpose of this book, then, is to provide an exposition to numerical relativity that is accessible, does not rely on an intimate prior knowledge of general relativity, and instead builds on familiar physics and intuitive arguments. Our hope is that it will help users of numerical relativity codes with limited expertise in general relativity “look under the hood”, allowing them to take a glance at what lies inside the codes and algorithms and to gain a level of familiarity with numerical relativity, so that they can use these codes “on the road” with greater confidence, understanding, and expectation of reliability.

One of us (TWB) has taught at several summer schools on numerical relativity and relativistic astrophysics over the last few years, and this experience has certainly helped to motivate our approach. The students, many of whom had limited experience with general relativity, were best served by skipping over some derivations and technical details and spending more time on motivating the physical and geometrical meaning of the variables and equations encountered in a 3+1 decomposition of Einstein’s field equations. This decomposition involves splitting four-dimensional spacetime into three-dimensional space plus one-dimensional time and recasting the equations accordingly. A brief introduction to general relativity is required first, of course, and we provide one here, together with an introduction to tensors and elements of differential geometry. However, since many concepts encountered in the 3+1 decomposition of the equations can be introduced in the much more familiar context of electrodynamics, it is possible to disentangle many issues associated with this 3+1 splitting from the special subtleties associated with general relativity, prior to discussing the latter. This is the approach we adopt in this volume.

Our goal, then, is neither to provide a comprehensive review of the entire field nor to present every derivation in complete detail; we essentially did just that in Baumgarte and Shapiro (2010). Instead, here we hope to provide a more accessible introduction to some concepts
and quantities encountered in numerical relativity, leaning heavily on analogies with Newtonian mechanics, scalar fields, and electrodynamics. The hope is that readers with a solid background in basic mechanics, electrodynamics, special relativity, and vector calculus, but with no familiarity, or only limited familiarity, with general relativity and differential geometry, will be able to follow the development. They should be able to gain an intuitive understanding of the concepts and variables arising in numerical relativity codes, recognize the structure of the equations encountered in a 3+1 decomposition, and appreciate some techniques often employed in their solution. Ideally, such a reader will then be motivated and able to consult any book in the list above for a more detailed and comprehensive discussion of one or more topics.

One of us (SLS) used an early draft of this book to teach a seminar on numerical relativity to a small group of junior and senior physics majors with little formal background in general relativity. They all seemed to warm to the approach and were able to absorb the basic ideas, as demonstrated by our weekly discussions and by their solving many of the exercises to be found throughout the volume.

Another feature that distinguishes this book from other textbooks on numerical relativity is that we include actual numerical codes. The broad availability of Python and its many libraries makes it easier than in the past to provide small sample scripts that, despite being limited in scope, highlight some features of many numerical relativity codes. We provide two such codes that illustrate the two basic types of problems encountered in numerical relativity: initial-value and evolution problems. These codes allow the reader to watch and explore some common computational algorithms “at work”.

This volume is organized as follows. In Chapter 1 we sketch Einstein’s theory of general relativity, relying extensively on comparisons with Newtonian gravity. Clearly, a single chapter on general relativity cannot possibly replace a full course or book on the subject, but we nevertheless hope that it will provide a quick review for readers with some knowledge of general relativity and, more importantly, simultaneously serve as a brief introduction and overview for readers unfamiliar with the subject.

In Chapter 2 we discuss how Einstein’s field equations in standard form can be recast to treat an initial-value problem in general relativity that can then be solved numerically. This step employs the 3+1 decomposition of spacetime that we mentioned above. We draw heavily on analogies with both scalar fields and electrodynamics, using these
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more familiar settings to introduce many of the geometrical objects encountered in numerical relativity. The result is a split of the field equations into constraint and evolution equations.

In Chapter 3 we introduce some strategies for solving the constraint equations, including conformal decompositions. Solutions to the constraint equations describe the fields at one instant of time and serve as initial data for dynamical evolutions.

In Chapter 4 we then discuss evolution calculations. This entails finding formulations of the evolution equations and coordinate conditions that result in stable numerical behavior.

In Chapter 5 we review simulations of binary black-hole mergers in vacuum as an example of such calculations. Simulations like those summarized here have played a crucial role in the detection and analysis of the gravitational waves mentioned above.

This book also includes several appendices. Appendix A summarizes some basic tensor concepts, including the covariant derivative.

Appendix B introduces some computational techniques commonly used in numerical relativity. In particular, this appendix includes the two Python scripts to which we referred above; one solves the (elliptic) constraint equations of Chapter 3 for an isolated black hole, and the other solves the (hyperbolic) evolution equations of Chapter 4 for a propagating electromagnetic wave. Both programs can also be downloaded from www.cambridge.org/NRStartingFromScratch. Here readers can vary the input parameters, run and modify the codes, and evaluate the output using the diagnostic graphing and animation routines that are also provided.

In the interest of providing a coherent development while keeping this volume short, we focus chiefly on vacuum spacetimes, i.e. those containing black holes and/or gravitational waves but no matter or other nongravitational fields as source terms in the Einstein equations. Stated differently, we concentrate on the left-hand side of Einstein’s equations (the “geometry”), setting the right-hand side (the “matter and nongravitational fields”) to zero. However, in Appendix C we extend our treatments of scalar and electromagnetic fields to identify their stress–energy tensors, which can serve as sources for Einstein’s equations, and show how general relativity imposes the familiar equations of motion for these nongravitational fields.

Appendix D provides a short summary of some of the most important mathematical results in differential geometry, general relativity, and the 3+1 decomposition that are encountered in numerical relativity.
Preface

For pedagogical purposes we have inserted a total of 74 exercises scattered throughout this book, some of which are numerical. They are designed to help readers test and develop their basic understanding of the material. In some cases the exercises build on each other, so in Appendix E we provide answers to those that are needed to tackle subsequent exercises.

We are deeply indebted to more people than we could possibly list here, including all those colleagues, teachers, and students from whom, over the years and decades, we have learned almost everything we know. Some of these have also had a direct impact on this book: We would like to thank Parameswaran Ajith, Andreas Bauswein, Stephan Rosswog, and Bjoern Malte Schaefer for inviting TWB to lecture at their summer schools, as well as the students at those schools for their feedback. We would also like to thank Elizabeth Bennewitz, Béatrice Bonga, Kenneth Dennison, Steven Naculich, and Constance Shapiro, who read sections of this book and provided helpful comments, and Eric Chown and Zachariah Etienne for their help with the Python scripts. Several reviewers gave us valuable feedback; we would like to thank Ulrich Sperhake in particular for his detailed report and helpful suggestions. We would like to express our gratitude to the seminar participants at the University of Illinois – Michael Mudd, Kyle Nelli, Minh Nguyen, and Samuel Qunell – who provided invaluable feedback on an early draft of this volume, as well as solutions to most of the exercises. We also gratefully acknowledge the National Science Foundation (NSF), the National Aeronautics and Space Administration (NASA), and the Simons Foundation for funding our research. Many figures in this volume were produced using Mayavi software. Our acknowledgements would not be complete, however, without thanking our wives and families for their enduring and loving support.

Finally, we hope we will be forgiven for referring quite often to our previous textbook throughout this volume for further details on various topics. As we are most familiar with that book, it is expedient for us to do so and we hope that other authors will not feel in any way slighted by our taking advantage of our previous exposition.

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