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Vadim Gorin is a faculty member at the University of Wisconsin–Madison and a member of the Institute for Information Transmission Problems at the Russian Academy of Sciences. He is a leading researcher in the area of integrable probability, and has been awarded several prizes, including the Sloan Research Fellowship and the Prize of the Moscow Mathematical Society.

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Preface

These are lecture notes for a one-semester class devoted to the study of random tilings. It was 18.177 taught at the Massachusetts Institute of Technology during the spring of 2019. The brilliant students who participated in the class,¹ Andrew Ahn, Ganesh Ajjanagadde, Livingston Albritten, Morris (Jie Jun) Ang, Aaron Berger, Evan Chen, Cesar Cuenca, Yuzhou Gu, Kaarel Haenni, Sergei Korotkikh, Roger Van Peski, Mehtaab Sawhney, and Mihir Singhal, provided tremendous help in typing the notes.

Additional material was added to most of the lectures after the class was over. Hence, when using this review as a textbook for a class, one should not expect to cover all the material in one semester; something should be left out.

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¹ In alphabetical order by last name.