1

Geometric Braids

The goal of this chapter is to define and begin to study braids in a mathematical sense. For this, in Section 1.1, we model a material braid by a family of arcs in $\mathbb{R}^3$, and introduce for every $n$ a space of $n$-strand braids whose elements are the equivalence classes of geometric braids with respect to one of several natural notions of deformation. In Section 1.2, we project the geometric braids onto planar diagrams encoded by certain words (the ‘braid words’), and are led to the braid isotopy problem, that is, recognizing if two diagrams, or the words that encode them, represent the same braid. In Section 1.3, we observe the failure of a few naive attempts to resolve the isotopy problem, calling for the development of more elaborate approaches.

Important note. The option retained throughout this text is to fully prove the results stated. Those of this chapter are in general natural and comprehensible, but their proofs often require a somewhat heavy geometric formalism. May the readers keep from becoming discouraged, and continue to resolutely plough forward, even if they skip over a few details, especially between Definition 1.1.11 and Section 1.2.2. Once the bases are mastered, the rest of the study, and in particular the algebraic arguments, will no longer pose this type of difficulty.

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1.1.1 From Material Braids to Geometric Braids

We propose here to elaborate a theory of braids. At the start, braids are material objects, such as braids of hair, or the sculpture of Figure 1.1: strands that cross each other while conserving a certain constant general orientation, but without forming knots.
Figure 1.1 An ornamental braid in salt dough.

Building a theory for such objects begins with a modelization: we need to decide which aspects to analyse and how to formalize material objects as mathematical objects susceptible to being studied. For the present case, we will neglect the metrical aspects, hence anything concerning the thickness of the strands, their length, their spacing, their curvature... and only retain the topological aspects, that is, the way the strands cross each other, whichstrand passes over or under another, etc. The theory we will develop is thus first and foremost a theory of crossings.

Since the metrical parameters of the strands are ignored, it is natural to modelize them by curves in the space $\mathbb{R}^3$. In a material braid, the strands have a beginning and an end, and are not broken along the way. We thus consider fragments of continuous curves, referred to as *arcs* in what follows. Consequently, we modelize an $n$-strand braid by the union of $n$ arcs. And since the strands cross each other, but without cutting each other, we require the arcs to be *pairwise disjoint*.

Next, as we are thinking of finite material braids, it is natural to suppose that the $n$ arcs emanate from $n$ fixed points in the plane $(0)\times\mathbb{R}^2$, for example $(0,1,0),..., (0,n,0)$ and terminate on $n$ fixed points in the plane $(1)\times\mathbb{R}^2$, for example $(1,1,0),..., (1,n,0)$.

Finally, we wish to exclude knots. We thus prohibit the arcs to turn back on themselves, hence conserving a constant general direction. For an arc $y$ joining the plane $(0)\times\mathbb{R}^2$ to the plane $(1)\times\mathbb{R}^2$, this condition reduces to requiring $y$ to be traced in $[0,1]\times\mathbb{R}^2$ and its intersection with every plane $(x)\times\mathbb{R}^2$ for $0 \leq x \leq 1$ to be a single point: if $y$ turned back on itself, there would exist a plane $(x)\times\mathbb{R}^2$ cutting $y$ in at least three points.

With this in mind, we are led to the following notion:

1 At least at first; we could after all imagine infinite braids...
1.1 The Geometric Braid Space

**Definition 1.1.1** (Geometric braid) An *n*-strand geometric braid is a union $\beta$ of $n$ arcs in $\mathbb{R}^3$, pairwise disjoint, linking the $n$ points $(0, i, 0)$, $i = 1, \ldots, n,$ to the $n$ points $(1, i, 0)$, $i = 1, \ldots, n,$ within the band $[0, 1] \times \mathbb{R}^2$, and whose intersection with every plane $\{x\} \times \mathbb{R}^2$ contains exactly $n$ points. The family of *n*-strand geometric braids is denoted $\mathcal{GB}_n$.

Before we go any further, note that an *n*-strand geometric braid can be naturally parametrized by a sequence of $2n$ functions from $[0, 1]$ into $\mathbb{R}$ or, if you prefer, a sequence of $n$ functions from $[0, 1]$ into $\mathbb{R}^2$. Indeed, any arc in $[0, 1] \times \mathbb{R}^2$ can be parametrized by the functions specifying its three coordinates. If this arc cuts each plane $\{x\} \times \mathbb{R}^2$ in one unique point, we can choose the abscissa as the parameter, and hence adapt a parametrization of the form $t \mapsto (t, f(t), g(t))$.

**Definition 1.1.2** (Parametrization of $\beta[i]$) If $\beta$ is an *n*-strand geometric braid, denote $\beta[i]$ the continuous function from $[0, 1]$ into $\mathbb{R}^2$ such that the arc emanating from $(0, i, 0)$ (‘the *i*th strand’ of $\beta$) is parametrized by the function $t \mapsto (t, \beta[i](t))$.

By definition, $\beta[i](t)$ is the tuple formed by the ordinate and the point of intersection of the *i*th strand of $\beta$ with $\{t\} \times \mathbb{R}^2$, and is thus determined by $\beta$. Conversely, $\beta$ is the set of points $\beta[i](t)$ where $(i, t)$ runs across $\{1, \ldots, n\} \times [0, 1]$. Hence, speaking of $\beta$ is purely and simply equivalent to speaking of the sequence $\beta[1], \ldots, \beta[n]$.

To conclude these preliminaries, note that the geometric braids can be equipped with a natural notion of distance. For this, we state that two braids...
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β, β' are close if the strands of β' remain close to the corresponding strands of β.

Definition 1.1.3 (Distance) For β, β' in GB_n, define

\[ d(β, β') := \sup\{∥β^{(i)}(t) - β'^{(i)}(t)∥ | i = 1, ..., n, t ∈ [0, 1]\}, \tag{1.1} \]

where ∥-∥ is the usual Euclidean norm on R^2.

Then d is a distance on GB_n (verify this!), and the associated topology coincides with that induced by the inclusion of GB_n in the space C([0, 1], R^{2n}) of continuous functions from [0, 1] to R^{2n}. Note that, since [0, 1] is a compact space, the supremum expressed in (1.1) is finite, and attained for at least one point.

1.1.2 Homotopy and Isotopy: Definitions

We have not yet finished with the modelization phase, and now tackle its most delicate aspect. We would like to elaborate a theory of braids retaining only their topological aspects. It is thus natural to consider as equivalent geometric braids that are, in some suitable manner, topologically indiscernible; this is what we would like to now formalize.

We might declare geometric braids equivalent when, as subspaces of R^3, they are homeomorphic. This idea quickly fizzles out: as a topological space, every n-strand geometric braid is homeomorphic to the union of n disjoint copies of the interval [0, 1] and hence, two geometric braids are always homeomorphic. The problem is clear: it is not a question of considering a geometric braid β as an abstract space, but as an embedding in [0, 1] × R^2, as specified for example by the functions β^{(i)}.

We thus seek to translate the idea of a deformation transforming one braid embedded in R^3 into another. Several formalizations will be introduced; we then prove their equivalence. The first exploits the notion of a path in the space GB_n. If X is a topological space, and a and b points of X, a path from a to b in X is a continuous mapping φ from [0, 1] into X satisfying φ(0) = a and φ(1) = b. We saw with Definition 1.1.3 that the geometric braid space GB_n can be equipped with a distance, hence with a notion of continuity, and thus we can speak of a path in GB_n.

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2 i.e. the norm defined by ∥(y, z)∥ := \(\sqrt{y^2 + z^2}\).

3 i.e. there exists a continuous bijection with continuous inverse (a ‘homeomorphism’) mapping one to the other.
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Definition 1.1.4 (Homotopic braids) Two geometric braids $\beta, \beta'$ of $GB_n$ are said to be homotopic, denoted $\beta \simeq \beta'$, if there exists a path from $\beta$ to $\beta'$ in $GB_n$.

As paths can be reversed and concatenated, homotopy is an equivalence relation on $GB_n$.

Example 1.1.5 (One-strand braids) Let $\beta, \beta'$ be two geometric braids each with a single strand. For $0 \leq s \leq 1$, let $\phi(s)$ be the geometric braid whose intersection with the plane $\{(t) \times \mathbb{R}^2\}$ is the barycentre of $\beta(t)$ and $\beta'(t)$, weighted with coefficients $1-s$ and $s$ (Figure 1.3). In other words, by identifying $\beta$ with $\beta^{[1]}$, we have $\phi(s) := (1-s)\beta + s\beta'$. Hence $\phi$ is a path from $\beta$ to $\beta'$ in $GB_1$, and $\beta$ and $\beta'$ are homotopic: there is only one single homotopy class in $GB_1$.

Note that the result does not extend to $GB_n$ for $n \geq 2$: mimicking the interpolation above with several strands at the same time could give arcs that cut each other, hence outside of $GB_n$.

The notion of homotopy of Example 1.1.5 is local in the sense that we in fact specify a deformation of the embedded braid, but without a deformation of the whole of the ambient space $[0,1] \times \mathbb{R}^2$. For a global approach, we can use the homeomorphisms of $[0,1] \times \mathbb{R}^2$. However, as above, demanding the existence of a homeomorphism of $[0,1] \times \mathbb{R}^2$ sending $\beta$ onto $\beta'$, where $\beta$ and $\beta'$ are geometric braids, is not a good idea, as we lose the idea of a continuous deformation\(^4\) of $\beta$ to $\beta'$. Nonetheless, we can amend the idea thanks to the notion of isotopy between homeomorphisms. For any space $\mathcal{X}$, denote $\text{Homeo}(\mathcal{X})$ the set\(^5\) of homeomorphisms of $\mathcal{X}$.

\(^4\) Think about the symmetry with respect to the plane $(1/2) \times \mathbb{R}^2$: it is indisputably a homeomorphism of $[0,1] \times \mathbb{R}^2$, but there is no reason at all for us to be able to continuously deform a figure onto its image by symmetry...

\(^5\) It is in fact a group when equipped with composition.
**Definition 1.1.6 (Isotopy)** Two homeomorphisms $\phi, \phi'$ of a topological space $X$ are said to be isotopic if there exists a path joining $\phi$ to $\phi'$ in $\text{Homeo}(X)$.

The idea is then to consider two geometric braids $\beta, \beta'$ equivalent if there exists a homeomorphism $\phi$ of $[0,1] \times \mathbb{R}^2$ sending $\beta$ onto $\beta'$ (considered as subsets of $[0,1] \times \mathbb{R}^2$) such that we can pass continuously from the identity to $\phi$, that is, that $\phi$ is isotopic to $\text{id}$. There is an essential condition: we only consider the homeomorphisms that are trivial on the two vertical boundary planes, $[0] \times \mathbb{R}^2$ and $[1] \times \mathbb{R}^2$. However, if $\Phi$ is an isotopy linking $\text{id}$ to $\phi$, the image of a braid by the homeomorphisms $\Phi(s)$ for $0 < s < 1$ has no reason to be a braid, since the images of the vertical planes $\{t\} \times \mathbb{R}^2$ have no reason to be vertical planes.\(^6\) Two notions, a priori distinct, emerge.

**Definition 1.1.7 (Isotopic braids)**

(i) A homeomorphism of $[0,1] \times \mathbb{R}^2$ is said to be trivial on the boundary if it leaves $[0] \times \mathbb{R}^2$ and $[1] \times \mathbb{R}^2$ invariant point by point. It is said to be stratified if it is trivial on the boundary and, in addition, leaves each vertical plane $\{t\} \times \mathbb{R}^2$ globally invariant; denote $\text{Homeo}^\text{st}([0,1] \times \mathbb{R}^2)$ and $\text{Homeo}^\text{nr}([0,1] \times \mathbb{R}^2)$ the two families thus formed.\(^7\)

(ii) Two geometric braids $\beta, \beta'$ are said to be isotopic (resp., isotopic in the unrestrained sense), denoted $\beta \approx \beta'$ (resp., $\beta \approx^\text{nr} \beta'$), if there exists a homeomorphism sending $\beta$ onto $\beta'$ and linked to $\text{id}$ by a path in $\text{Homeo}^\text{nr}([0,1] \times \mathbb{R}^2)$ (resp., in $\text{Homeo}^\text{st}([0,1] \times \mathbb{R}^2)$).

The relation $\beta \approx \beta'$ implies $\beta \approx^\text{nr} \beta'$: a path in the subspace $\text{Homeo}^\text{st}([0,1] \times \mathbb{R}^2)$ of $\text{Homeo}^\text{nr}([0,1] \times \mathbb{R}^2)$ is, in particular, a path in $\text{Homeo}^\text{st}([0,1] \times \mathbb{R}^2)$. Moreover, since the isotopies between homeomorphisms can be composed or reversed, the relations $\approx$ and $\approx^\text{nr}$ are equivalence relations.

Let us recapitulate. Three equivalence relations have been introduced to modelize the idea of the deformation of a geometric braid, all expressed in terms of the existence of a continuous path linking two braids. With the homotopy $\approx^h$, we only consider the strands of the braids, independently from the ambient space; in contrast, with the isotopies, we stipulate the existence of a deformation of the entire ambient space (in fact, we often speak of an ‘ambient isotopy’). The default version will here be $\approx$: restricting ourselves to stratified homeomorphisms might seem artificial, but this is the condition to remain within the braid space, which is our reference framework. It is nevertheless legitimate to ask whether removing this restriction would change things, and this is why the unrestrained version was introduced. We will have to wait for

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\(^6\) Nor even regular surfaces, as no differentiability condition has ever been mentioned.

\(^7\) The symbol $\partial$ is commonly used to represent the boundary of a space.
Chapter 5 for all the links to be clarified, but the response will be optimal: the
relations \( \approx^h \), \( \approx \), and \( \approx^{nr} \) are equivalent, thus leading to a unique, natural, and
robust notion.

### 1.1.3 Homotopy and Isotopy: Equivalence

Indeed, the three notions above are equivalent, providing both legitimacy to the
unique notion thus exposed, and a technical convenience, as we can use one
version or another as convenient. In this section, we establish the equivalence
between the first two notions.

**Proposition 1.1.8 (Equivalence)** Two geometric braids are isotopic if and
only if they are homotopic.

The case of unrestrained isotopy is left aside for the moment. It will be
treated in Chapter 5.

One direction of the equivalence is intuitive: in an isotopy, ignoring the
exterior of the braids leads to a homotopy.

**Lemma 1.1.9** Two isotopic geometric braids are homotopic.

**Proof** Let \( \beta, \beta' \) be braids in \( GB_n \) and suppose that \( \Phi \) is an isotopy attesting
\( \beta \approx \beta' \), hence satisfying
\[ \Phi(0) = id \text{ and } \beta' = \Phi(1) \circ \beta. \]
For \( s \) in \([0,1]\), set
\[ \phi(s) := \Phi(s) \circ \beta. \]
By definition, \( \Phi(s) \) sends any geometric braid to another
geometric braid and hence, in particular, \( \phi(s) \) belongs to \( GB_n \). The continuity
of \( \Phi \) implies that of \( \phi \). Finally, we have
\[ \phi(0) = \Phi(0) \circ \beta = \beta \quad \text{and} \quad \phi(1) = \Phi(1) \circ \beta = \beta'. \]
Hence \( \phi \) is a path linking \( \beta \) to \( \beta' \) in \( GB_n \), showing \( \beta \approx^h \beta' \).

The other direction requires more care, but is not terribly difficult. We begin
with single-stranded braids, which we have seen in Example 1.1.5 are always
homotopic.

**Lemma 1.1.10** Two one-stranded geometric braids are isotopic.

**Proof** Let \( \beta, \beta' \) be two geometric braids on one strand.\(^9\) We start with the
path \( \phi \) linking \( \beta \) to \( \beta' \) in \( GB_1 \) constructed in Example 1.1.5. We seek a path \( \Phi \)
in \( \text{Homeo}^d([0,1] \times \mathbb{R}^2) \) extending \( \phi(s) \) in the sense where, for every \( t \), it sends

\[^8\] Writing \( \beta' = \Phi(1)(\beta) \) is tempting, but not formally correct: as \( \beta \) is identified with a family of \( n \)
mappings from \([0,1]\) into \([0,1] \times \mathbb{R}^2\), this says that these mappings composed with \( \Phi(1) \) are
equal to those constituting \( \beta' \).

\[^9\] As in Definition 1.1.4, we identify \( \beta \) with the function \( \beta^{[1]} \) of \([0,1]\) into \( \mathbb{R}^2 \).
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β(t) to φ(s)(t), that is, to \((1−s)β(t)+sβ′(t)\). For this it suffices to choose the translation that does the work, namely

\[
Φ(s)(x, y, z) := (x, y, z) + s(0, \bar{β}(x)\bar{β}'(x)). \tag{1.2}
\]

By construction, Φ(s) is a stratified homeomorphism of \([0, 1]×\mathbb{R}^2\) for every s. Moreover, Φ(0) is the identity, and Φ(1) sends β onto β′ since, for every x, we find Φ(1)(β(x)) = β(x) + β(x)β′(x) = β′(x). Consequently, Φ attests that β and β′ are isotopic.

For braids with at least two strands, it is in general impossible to find a translation sending simultaneously each of the n points of the intersection of β with a plane \(\{x\}×\mathbb{R}^2\) to where we wish them to go, and we must find a more precise construction. For this, we consider the homeomorphisms trivial outside of a neighbourhood of the strands of the braids under consideration.

**Definition 1.1.11** (Tubular neighbourhood)

(i) For a in \(\mathbb{R}^2\) and \(ρ > 0\), denote \(D(a, ρ)\) the open disk with centre a and radius ρ in \(\mathbb{R}^2\).

(ii) For β in \(\mathcal{GB}_n\) and ρ > 0, the ρ-tubular neighbourhood of β, denoted \(V(β, ρ)\), is the open subset of \([0, 1]×\mathbb{R}^2\) whose intersection with the plane \(\{x\}×\mathbb{R}^2\) is the union of the n disks \(D(β(x), ρ)\).

Thus, \(V(β, ρ)\) is a union of open tubes, each surrounding one of the strands of β with a radius ρ. Note that, if β and β' are n-strand geometric braids, then \(d(β, β') < ρ\) if and only if, as a set of points, β' is contained in \(V(β, ρ)\).

We can thus amend the result of Lemma 1.1.10 to find a trivial isotopy \(^{10}\) outside of a neighbourhood of the initial braid.

**Lemma 1.1.12** If β,β′ are two one-strand geometric braids satisfying \(d(β, β') < ρ\), then β and β′ are isotopic via an isotopy Φ such that, for any s, the homeomorphism Φ(s) is trivial outside of \(V(β, ρ)\).

**Proof** As in Lemma 1.1.10, we retain the idea of using the translation in the plane \(\{x\}×\mathbb{R}^2\) that sends \(β(x)\) onto \(β′(x)\), but modulus a coefficient tending to 0 on the boundary of the disk \(D(β(x), ρ)\). For this, if \(D\) is a disk in \(\mathbb{R}^2\) with centre \((y_0, z_0)\) and radius \(ρ\), denote \(Λ_D\) the function from \(\mathbb{R}^2\) into \(\mathbb{R}\) defined by \(Λ_D(y, z) := 0\) for \((y, z) \not∈ D\) and

\[
Λ_D(y, z) := 1 - \frac{1}{ρ}∥(y - y_0, z - z_0)∥ \quad \text{for} \quad (y, z) ∈ D.
\]

Thus, \(Λ_D\) takes on the value 1 at the centre of \(D\) and decreases linearly with

\(^{10}\) i.e. coincident with the identity mapping.
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the distance from the centre to be zero on the boundary and outside: its graph
is a Chinese hat on top of \((y_0, z_0)\). Note that \(\Lambda_D\) depends continuously on
the parameters \(y_0, z_0\), and \(\rho\), and for any \((y, z)\) and \((y_1, z_1)\) inside \(D\), the translated
point \((y, z) + \Lambda_D(y, z)(y_1 - y_0, z_1 - z_0)\) is inside \(D\).

Returning to the braids \(\beta\) and \(\beta'\), we define \(\Phi\) by

\[
\Phi(s)(x, y, z) := (x, y, z) + s \Lambda_{D(\beta^{[1]}(x), \rho)}(y, z)(0, \beta^{[1]}(x)\beta'^{[1]}(x)).
\] (1.3)

Then, \(\Phi\) is a continuous function of \([0, 1]\) into \(\text{Homeo}^d([0, 1] \times \mathbb{R}^2)\). Moreover, \(\Phi(0)\) is the identity, and \(\Phi(1)\) sends \(\beta\) onto \(\beta'\) since, by construction, \(\Lambda_{D(\beta^{[1]}(x), \rho)}(\beta'^{[1]}(x))\) is equal to 1 for every \(x\). Furthermore, \(\Lambda_{D(\beta^{[1]}(x), \rho)}\) is zero outside of \(D(\beta^{[1]}(x), \rho)\), hence \(\Phi(s)\) is trivial outside of \(V(\beta, \rho)\). Finally, the convexity of \(D(\beta^{[1]}(x), \rho)\) guarantees that, for every \(s\), the point \((1 - s)\beta^{[1]}(x) + s\beta'^{[1]}(x)\) is in \(D(\beta^{[1]}(x), \rho)\) since, by hypothesis, so is \(\beta'^{[1]}(x)\).

Extending the construction to the case of several strands is simple as long as the
distance between the two braids is sufficiently small: it suffices to enclose each strand in an appropriate tubular neighbourhood and to take the sum of the associated tiny translations. The point is that for a geometric braid, two strands can never be arbitrarily close.

**Definition 1.1.13 (Minimal strand-spacing)** The minimal strand-spacing of an \(n\)-strand geometric braid \(\beta\) is defined by

\[
e(\beta) := \frac{1}{2} \inf \{ \|\beta^{[1]}(t) - \beta'^{[1]}(t)\| \mid 0 \leq t \leq 1, \ 1 \leq i < j \leq n \}.
\]

For any \(i\), the domain of definition of \(\beta^{[i]}\) is the compact interval \([0, 1]\), hence every function \(t \mapsto \|\beta^{[1]}(t) - \beta'^{[1]}(t)\|\) attains its minimum. This must be strictly positive, as \(\beta^{[1]}(t) = \beta'^{[1]}(t)\) would mean that the \(i\)th and \(j\)th strands of \(\beta\) cross each other in the plane \(\{t\} \times \mathbb{R}^2\). The minimal strand-spacing is thus always strictly positive.

Applying the attenuated translations of Lemma 1.1.12, we deduce that two braids sufficiently close are always isotopic.

**Lemma 1.1.14** Two geometric braids \(\beta, \beta'\) satisfying \(d(\beta, \beta') < e(\beta)\) are isotopic.

**Proof** Suppose that \(\beta\) and \(\beta'\) have \(n\) strands, and set

\[
\Phi(s)(x, y, z) := (x, y, z) + s \sum_{i=1}^{\infty} \Lambda_{D(\beta^{[1]}(x), e(\beta))}(y, z)(0, \beta^{[1]}(x)\beta'^{[1]}(x)).
\]

This barbaric formula is in fact quite simple: \(\Phi(s)\) does nothing far from the strands of \(\beta\), whereas in the neighbourhood of the \(i\)th strand of \(\beta\), it performs a...
small ‘attenuated’ translation in the direction of $\beta_i^0(x)$ to $\beta'_i(x)$. The hypothesis that the strands of $\beta$ are never at a distance less than $2e(\beta)$ guarantees that the translations take place in the interior of tubes surrounding the strands of $\beta$; these tubes are disjoint by the definition of $e(\beta)$. In particular, $\Phi(s)$ sends every geometric braid onto a geometric braid. Then $\Phi(0)$ is the identity for $s = 0$, and $\Phi(1)$ sends every point $(t, \beta^0(t))$ to the corresponding point $(t, \beta'^0(t))$: Hence $\Phi$ attests that $\beta$ and $\beta'$ are isotopic.

Note that, as in Lemma 1.1.12, the above isotopy linking $\beta$ to $\beta'$ is trivial outside the tubular neighbourhood $V(\beta, e(\beta))$.

It is now easy to establish the converse of the implication of Lemma 1.1.9 and obtain the equivalence of Proposition 1.1.8.

**Proof of Proposition 1.1.8** The implication ‘isotopic $\Rightarrow$ homotopic’ was seen in Lemma 1.1.9. Conversely, suppose that $\phi$ is a path linking $\beta$ to $\beta'$ in $\mathcal{GB}_n$. Define $\beta_s$ to be $\phi(s)(\beta)$ (hence $\beta_0 = \beta$ and $\beta_1 = \beta'$).

If ever we have $d(\beta, \beta') < e(\beta)$, then $\beta$ and $\beta'$ are isotopic by Lemma 1.1.14. This condition has no reason in general to be satisfied, but we will reduce to it by sectioning the path $\phi$. First, the function $s \mapsto e(\beta_s)$ of $[0, 1]$ into $\mathbb{R}_{>0}$ is continuous and defined on a compact set, hence it attains its minimum $\varepsilon$, strictly positive. By definition, we have $e(\beta_s) \geq \varepsilon > 0$ for all $s$. Next, by hypothesis, $\phi$ is a continuous function of the compact space $[0, 1]$ in the metric space $(\mathcal{GB}_n, d)$. It is thus uniformly continuous and, consequently, there exists $\delta$ such that

$$|s' - s| < \delta \quad \text{implies} \quad d(\beta_{s'}, \beta_s) < \varepsilon. \quad (1.4)$$

Let $s_0, ..., s_p$ be real numbers satisfying $s_0 = 0$, $s_p = 1$, and $s_j < s_{j+1} < s_j + \delta$ for all $j$. Consider the braids $\beta_{s_0}, ..., \beta_{s_p}$. For every $j$, Equation (1.4) implies $d(\beta_{s_j}, \beta_{s_{j+1}}) < \varepsilon$, hence $d(\beta_{s_j}, \beta_{s_{j+1}}) < e(\beta_{s_j})$. Then Lemma 1.1.14 implies that, for every $j$, the braids $\beta_{s_j}$ and $\beta_{s_{j+1}}$ are isotopic. By the transitivity of isotopies, $\beta_{s_0} = \beta$ is isotopic to $\beta_{s_p} = \beta'$.

### 1.1.4 The Braid Space

Since homotopy and isotopy coincide, there is no reason to differentiate the two equivalence relations on the spaces of geometric braids. In what follows, we will only speak of isotopy of geometric braids, and hence only use the symbol $\approx$. Nevertheless, homotopy remains important, as in practice, to show