Solving Problems of Simple Structural Mechanics

Solve problems in elementary structural mechanics thoughtfully and efficiently with this self-contained volume. Cover the basics of structural mechanics and focus on simple structures, truss frameworks, beams and frames, design choices and deformity. Carefully interrogate underlying assumptions for efficiencies in working out whilst expounding fundamental principles for a consistent understanding. Connect the practical world of indeterminate structures to their analysis, to underline benefits they impart to the latter: that certain analytical methods provide a wealth of efficient solutions for problems of indeterminate structures compared to determinate ones. Celebrate the beauty of analytical indeterminacy and its relationship to practical structures. Perfect for students invested in structural mechanics.

Keith Alexander Seffen leads internationally recognised research into shape-changing structures in the Advanced Structures Laboratory at Cambridge, which he co-founded.

Solving Problems of Simple Structural Mechanics

KEITH ALEXANDER SEFFEN
University of Cambridge

CAMBRIDGE
UNIVERSITY PRESS

CAMBRIDGE
UNIVERSITY PRESS

University Printing House, Cambridge CB2 8BS, United Kingdom

One Liberty Plaza, 20th Floor, New York, NY 10006, USA

477 Williamstown Road, Port Melbourne, VIC 3207, Australia

314–321, 3rd Floor, Plot 3, Splendor Forum, Jasola District Centre, New Delhi – 110025, India

103 Penang Road, #05–06/07, Visioncrest Commercial, Singapore 238467

Cambridge University Press is part of the University of Cambridge.

It furthers the University's mission by disseminating knowledge in the pursuit of education, learning, and research at the highest international levels of excellence.

www.cambridge.org
Information on this title: www.cambridge.org/9781108843812
DOI: 10.1017/9781108920131

First published 2022

A catalogue record for this publication is available from the British Library.

Library of Congress Cataloging-in-Publication Data
Names: Seffen, K. A. (Keith A.), author.
Title: Solving problems of simple structural mechanics / Keith Alexander Seffen, Department of Engineering, University of Cambridge, Trumpington Street, Cambridge.
Description: New York : Cambridge University Press, 2022. | Includes bibliographical references and index.
Identifiers: LCCN 2021017134 (print) | LCCN 2021017135 (ebook) | ISBN 9781108843812 (hardback) | ISBN 9781108920131 (epub)
Subjects: LCSH: Structural engineering–Mathematics.
Classification: LCC TA640 .S44 2021 (print) | LCC TA640 (ebook) | DDC 624.1–dc23
LC record available at https://lccn.loc.gov/2021017134
LC ebook record available at https://lccn.loc.gov/2021017135

ISBN 978-1-108-84381-2 Hardback

Cambridge University Press
978-1-108-84381-2 — Solving Problems of Simple Structural Mechanics
Keith Alexander Seffen
Frontmatter
More Information

Contents

Preface

This book is for students already invested in Structural Mechanics. They know about forces and moments, and couples from pairs of applied forces. They understand the concepts of equilibrium, compatibility and stiffness; of how beams and pin-jointed frameworks differ in their constituent behaviour; of the nature of supports; and of the concept of statical equivalency.

I reflect the usual gradation in complexity and form, moving from bodies to bars, to cables, and to beams, and sometimes, mixing them up. I consider different metrics of structural design: of safe loading, of failure by plastic collapse and buckling, of cross-sectional limitations and joint design, for example.

In particular, I focus on how to think about (and solve) ‘Structures’ problems *better*. Formulaic analysis methods are not overly represented because they can be applied without understanding fully how they work. Instead, I present a dialogue of how solving proceeds, collated as short chapters of worked examples – without the usual repetitive exercises at their ends.

I apologise for my greyscale figures. I am a child of the hard-copy age where colour was taxed, limiting that part of my presentation still. I dispense with denoting vectors in boldface because the direction of quantity is always implied. Parameters that vary are italicised, but labels are roman and upright; ‘A’ can be a point in an area ‘A’. Greek letters typically denote fundamental quantities or dimensionless groups, but not always.

Purpose

I consider mostly *statically indeterminate* (or indeterminate, alone) structures but not exclusively, since despite more solving effort (because with equilibrium there are also geometrical compatibility and material laws, our three *imperatives*, to include), the scope for more efficient, more confident solutions is broadened. Also, because most practical structures are indeterminate.

For example, indeterminacy is often couched in terms of extra statical unknowns from ‘too many’ members or supports. Furthermore, a determinate structure (and thus a soluble one by equilibrium alone) can be wrought by subtracting certain of their number from the original indeterminate structure, which we then label to be *redundancies*.

But indeterminacy is a function of how the structure is built *entirely*: no particular member or support naturally identifies as being redundant. We can rightly set *any* statical quantity to be redundant in our analysis, especially if less working out follows – when finding elastic displacements via *Virtual Work*, for example.

Many good textbooks, however, insist that the imperatives, conceptually, stand alone from each other, which is reinforced by their sequential employment during solution. That equilibrium only depends on the initial geometry despite the distortions which follow, with the material obediently furnishing a governing linear connection between them. Of course, adhering to small displacements is one reason.

But it is precisely because they are linked during solution that indeterminacy allows us to probe deeper into fundamental behaviour. For example (again), mechanistic and thus significant (and potentially disastrous) departures from initial geometry can combine favourably with redundancy, to suggest a new type of *non-linear* stiffening overall as distortions accrue; see Chapter 4.

During plastic design, material ductility is mandatory, in order that any viable equilibrium solution, *i.e.* one of our choosing, expresses a safe loading. Put another way, we can choose the *wrong* equilibrium solution in view of compatibility yet achieve a safe working, which is anathema to our sense of engineering precision.

Compensating for our apparent error is wrought by the action of indeterminacy, specifically, by the plastic deformation being able to *redistribute* elsewhere in the structure because its material *is* ductile. In trade-off, the cross-section has to increase in size which, however, increases the safety factor for our structure.

Indeterminacy thus cultivates analytical advantage and kinematical insight. It should not be portrayed as the bane of structural simplicity, where solving determinate cases is largely an instructive exercise. I hope my examples provide suitable demonstration and a different insight into solving Structures. Their content is now described.

Layout

The following chapter quickly revises some key concepts: geometry and distortion, the *generalised* Hooke's laws for bars and beams, and well-known energy methods. There are then 21 chapters in 6 parts.

Simple Structures

The first structures are rigid bodies acted upon by frictional forces in Chapter 1. All examples have one statical unknown; and when Columb's Inequality ($F \leq \mu N$) is satisfied at the point of slipping, each system becomes determinate. Their equilibrium geometry can also be solved by graphical means – more quickly so, for the examples chosen.

Chapter 2 deals with the displacements of rigid bodies supported on elastic springs or by fluid buoyancy. These reaction forces depend on the very displacements they induce, and the deformed geometry must feature to define them; in effect, these forces assume a constitutive character. Nevertheless, by maintaining small displacements, overall equilibrium can be assessed via the initial layout.

Geometry and equilibrium come together again in Chapter 3 for loaded cables, which are infinitely flexible; their deformed equilibrium is governed similarly to beams.

Truss Frameworks

The next three chapters (Part II) deal with elastic truss frameworks.

We deploy *Maxwell's Rule* in Chapter 4 to calculate the number of redundancies. This rule is a statement of absolute rigidity, derived from counting the bars, joints and supports. A positive tally expresses the degree of indeterminacy, and *vice versa* for the number of mechanistic motions. For certain truss layouts, indeterminacy and mechanistic action can become conflated, where their stiffness 'emerges' only when there is deformation; Maxwell's Rule is modified accordingly.

Elastic bar extensions are calculated from the method of Virtual Work in Chapter 5. Because indeterminacy promotes infinitely many equilibrium sets of bar tensions, we can explore their variety in order to enhance the Virtual Work process. Several truss examples are presented, including one with *mis-fitting* bars.

Truss design by the Lower Bound method is considered in Chapter 6. The method is rarely applied to trusses, but in this example the exact *elasto-plastic* response is soluble, in order to highlight the method's efficacy, in particular the role played by the members' ductility in redistributing the loading capacity during permanent yielding of the material.

Beams and Frames: Character

Characterising beam and frame behaviour is then garnered in the three chapters of Part III.

Bending moment and shear force diagrams are their equilibrium signatures. In Chapter 7, we think about the loads themselves, especially about point loadings – in isolation; what happens *away* from loads is equally important for drawing diagrams properly. The sign convention, which commands little mention elsewhere, is strictly enforced at all times, and drawing diagrams becomes an holistic, assured exercise.

The level of redundancy in beams (and frames) is determined in Chapter 8 by *ad hoc* counting procedures based on inserting pins or cutting the structure. Furthermore, using symmetry (and anti-symmetry) concepts in Chapter 9, we can declare certain statical quantities to be zero; the arguments are also developed for kinematical quantities.

We limit ourselves to singly symmetrical/anti-symmetrical layouts, where the mid-line performance is key. Both halves of the structure are compared by *flipping* or *spinning*, where contradictory (and hence zero) parameters can be identified.

Beams and Frames: Analysis

The techniques above then contribute to our analysis of indeterminate structures (Part IV). Our analytical basis is the *Force Method*, which separates the structure into constituent determinate parts, sharing equal and opposite redundant forces and moments. Compatibility of their corresponding deflections furnishes a complete solution.

We therefore revise determinate beams and cantilevers in Chapter 10, making use of *polynomial functions* for their exact displacements. Particular layouts of initial geometry and loading lead to a list of *standard case* results, which underpin the method of *Deflection Coefficients* in Chapter 11 for solving indeterminate cases.

The stiffness of a simply-supported beam loaded by a couple is found in Chapter 12. Despite its determinate nature, the analysis is somewhat involved. The beam is then modified – and made redundant – by adding another roller support, which, importantly, is collocated with the couple. The new stiffness is much easier to calculate even though indeterminacy would prepare us for more effort. Furthermore, the bending moment performance around the new support enables us to think about the performance of a more detailed junction, of two or more beams meeting.

Design Choices

Making informed choices for the design of a structure then follow in Part V.

We recall first in Chapter 13 that pin-joints are an idealisation, yet many rigidly connected frameworks are treated as pin-jointed. We compare two identical layouts of a simple arch with rigid and pinned joints, loaded by point forces to promote axial forces as well as bending. As their member slenderness increases, we find a diminishing rigidity for all joints.

Chapter 14 introduces short-cuts for calculating the second moment of area for symmetrical cross-sections in which the neutral axis is the usual major (or minor) axis. When the applied bending moment is no longer parallel to either axis, the corresponding neutral axis is altogether different from the bending direction; calculating the elastic stiffness is no longer straightforward.

Optimal structural performance is then explored in two ways. First, at the level of cross-section, by comparing bending and torsional stiffness; the latter prepares us for analysis of ‘pseudo’ three-dimensional frames – *grillages*. Second, in terms of structural stiffness and strength from transverse loading alone, in order to highlight the ‘natural’ limits of cross-sectional proportions either way.

Establishing the strength of a cross-section takes place in Chapter 15. A Lower Bound approach permits any viable equilibrium solution, which may reduce our working with marginal sacrifice in the accurate strength. The interaction between multiple stress resultants also relates to joint detailing. Further Lower Bound demonstration for designing the diverse supports of multi-span beams is given in Chapter 16.

Analysis of indeterminate grillages is introduced in Chapter 17. Bending and torsion (and shearing) are automatically coupled because of out-of-plane loading, to a degree prescribed by the grillage layout and nature of its joints. We explore this coupling character by solving a variety of examples (using, in part, the method of Deflection Coefficients).

Deliberately Deformed

The role of deformed geometry is then celebrated in the final part – Part VI.

For collapse of an indeterminate beam or frame, we determine the least number of hinges required and their positions in Chapter 18, and thence the family of collapse modes. General collapse motions are calculated from the method of *Instantaneous Centres* for multiple, interacting loads; and the correct mechanism ultimately correlates with the best Lower Bound equilibrium solution.

Beam *buckling* in Chapter 19 focuses on the mathematical nature of the solution and the apparent paradox that displacements remain unresolved even though they must feature in the formulation; and that buckling for actual *imperfect* structures is a misnomer. Chapter 20 considers more elaborate buckling cases where the nature of the governing loads changes dramatically with deformation.

Finally, we consider the effect of heating a material upon the structural response in Chapter 21. The inclusion of temperature effects at a material level is straightforward, but the outcome structurally is more complex. The primary example is a bimetallic strip, whose performance *deliberately* celebrates displacements *increasing* when most structural design does not.

Acknowledgements

Writing a book about solving Structures problems inevitably echoes examples from other books; I aim, nevertheless, to express my own solutions – as this book purports. A few examples have been directly reworked from established texts and are referenced locally. Some examples have been taken directly, and inspired indirectly, from Part I of the Engineering Tripos at the University of Cambridge, with original figures redrawn and their solutions altogether reworked. Their inventors from the Structures Group are collectively credited but noting in particular Professors Chris Calladine (emeritus), Jacques Heyman (emeritus), Chris Burgoyne (emeritus), Simon Guest, Cam Middleton, Sergio Pellegrino (now at Caltech), Janet Lees, Allan McRobie, and Dr Chris Morley (retired). I have devised the rest either as part of my own Tripos lecturing duties or in writing this book. If I have been negligent in my credits (and gratitude) elsewhere, it has been unintentional.

Author's Note

Straining

Often we have to deal with relatively small changes in geometrical quantities because of simplifying assumptions. Consider, for example, a narrow right-angled triangle. We find the length of the hypotenuse to be $\sqrt{L^2 + a^2}$ where a is the smallest side-length; or $L\sqrt{1 + (a/L)^2}$.

We make further progress using the Binomial Theorem, which states,

$$(1 + x)^n = 1 + nx + \frac{n(n-1)}{2!} \cdot x^2 \ldots. \tag{0.1}$$

Thus, our hypotenuse length becomes

$$L\left[1 + \frac{1}{2}\left(\frac{a}{L}\right)^2 - \frac{1}{8}\left(\frac{a}{L}\right)^4 \ldots\right] \approx L\left[1 + \frac{1}{2}\left(\frac{a}{L}\right)^2\right] \tag{0.2}$$

when a is much smaller than L, and a/L much less unity: for $x \ll 1$, we may set $(1 + x)^n$ equal to $1 + nx$.

When a rigid bar of length L pivots in plane by a small angle θ about one end, the other experiences a small normal displacement $L\theta$. If the bar is inclined at a general angle α to the horizontal, the vertical and horizontal displacement components are $L\theta \cdot \cos\alpha$ and $L\theta \cdot \sin\alpha$, respectively: or, incidentally, $L\cos\alpha \cdot \theta$ and $L\sin\alpha \cdot \theta$, the same. The components multiply the absolute rotation by the *projections* of the bar length onto the required directions.

Now consider an elastic bar, which also extends by a small length e. The in-plane displacement components of one end relative to the other are now e and $L\theta$. Clearly, the axial strain of the bar, e/L, is unaffected by the rotation, no matter its size.

If the strain is computed instead from the change in bar length according to the displacement components of its end, we first observe a new length of $\sqrt{(L + e)^2 + (L\theta)^2}$. Using the Binomial Theorem and retaining terms up to second order, we have $L(1 + e/L + (1/2) \cdot [(e/L)^2 + \theta^2])$.

Discounting the much smaller squared terms sets the length to be $L(1+e/L)$, hence, our expected strain. Even though the orthogonal displacement components may be of similar size, the axial strain is garnered only from e, the component along the bar.

However, when e is negligible compared to $L\theta$, we can make two claims about the strain. Either there is no strain because there is only pure rigid body rotation or, if second-order terms are kept, the new length is $L(1 + (1/2) \cdot \theta^2)$, similar to the hypotenuse expression. There is thus an apparent axial strain equal to $\theta^2/2$.

This is not physically possible but arises analytically because $L\theta$ has been assumed to act normal to the original bar axis. Movement of the rotating end on a strict circular path also produces a second-order axial component of displacement, equivalent to an equal and opposite (and thence negating) strain compared to above.

These second-order effects in the kinematics of deformation are an example of *geometrical non-linearity*. They are, for most problems, negligible, but in certain cases they provide valuable insight when first-order effects are remiss. Invoking them, however, requires careful thought about how they are formulated, as we have just seen.

Curving

The curvature of a circle of radius R is obviously $1/R$; or $2\pi/2\pi R$, the total angle enclosed (or subtended) by the circle, divided by the circumference. More generally, a portion of circumference with arc-length s, equal to $R\theta$, sets $1/R = \theta/s$ and the curvature as the ratio of the local subtended angle to arc-length.

This definition can be contracted to elemental values when the curvature (and R) varies along the 'curve'. Denoting by κ:

$$\kappa = \lim_{\delta s \to 0} \frac{\delta\theta}{\delta s} = \frac{\mathrm{d}\theta}{\mathrm{d}s}. \tag{0.3}$$

When beams deflect, they engage displacements *transverse* to their original stress-free layout. Geometrical continuity demands curving of the beam, and thus displacements are related to curvature and curvature itself to the structural response of the beam. We should also imagine the beam to be slender, *i.e.* very thin compared to its length, as if condensed into a line for the beam axis.

An orthogonal coordinate system can be superposed conveniently onto an originally straight beam with x along and v normal for displacements. The gradient of the deflected shape is now $\mathrm{d}v/\mathrm{d}s$ along an *intrinsic* coordinate s on the displaced beam.

Over a small deflected element of original length δx, the displacements change by δv, and Pythagoras sets $\delta s^2 = \delta v^2 + \delta x^2$. Dividing through by δx, using the Binomial Theorem, and observing the limit, we find

$$\frac{\mathrm{d}s}{\mathrm{d}x} \approx 1 + \frac{1}{2}\left(\frac{\mathrm{d}v}{\mathrm{d}x}\right)^2. \tag{0.4}$$

When $\mathrm{d}v/\mathrm{d}x$ is small, the second term above is negligible compared to unity, setting $\delta s \approx \delta x$ and $\mathrm{d}v/\mathrm{d}s \approx \mathrm{d}v/\mathrm{d}x$. This is known as the shallow gradient assumption, which predicates small displacements for moderate length, slender beams.

We can now replace θ by $\mathrm{d}v/\mathrm{d}x$ and δs by δx in the original definition of curvature, Eq (0.3), to give

$$\kappa = \frac{\mathrm{d}}{\mathrm{d}x}\left(\frac{\mathrm{d}v}{\mathrm{d}x}\right) = \frac{\mathrm{d}^2 v}{\mathrm{d}x^2}. \tag{0.5}$$

The *direction* of curvature at a given position on the beam axis can be specified by where the centre of curvature lies relative to the axis, *i.e.* the origin of the local *radius* of curvature, much like the centre of our first circle.

For example, let the displacements be described by $v = Ax^2/2$, an upwardly curving parabola. Equation (0.5) returns a constant curvature, $\kappa = A$, and thus a centre of radius located $1/A$ *above* the original beam line. For this, our definition of positive curvature, Eq. (0.5) remains; if not, the right-hand side is multiplied by -1.

Stiffness and Hooke

At a material level, elastic stress is linearly related to strain by the Young's Modulus, E, through Hooke's Law. This is our *constitutive law* for *direct* behaviour, when the stress acts normal to an exposed cross-section.

Shearing, on the other hand, acts tangentially with an equivalent modulus, G. For isotropic materials, $G = E/2(1+\nu) \approx 0.38E$ when the Poisson's Ratio, ν, equals 0.3 (as in most Engineering metals).

Multiplying a uniformly distributed stress, σ, by a cross-sectional area, A, we have an axial force, F, say. Applying F as equal and opposite forces to the ends of an elastic bar, it extends relatively and axially by e. The axial strain, ϵ, is e/L given an unstressed bar length, L. From Hooke's Law, $\sigma = E\epsilon$:

$$\frac{F}{A} = E \cdot \frac{e}{L} \quad \rightarrow \quad F = \frac{EA}{L} \cdot e. \tag{0.6}$$

The force is in linear relation to the extension, with a constant of proportionality now equal to our bar's axial stiffness. Reflecting the character of constitutive behaviour at a structural level is said to express a *generalised* Hooke's Law.

Note that we have invoked three *imperatives*. We have an equilibrium statement $F = \sigma A$ and a compatibility statement $e = \epsilon L$, bound together by the original material Hooke's Law. The internal force represents an aggregated stress performance, or a *stress resultant*. The uni-dimensional nature of Eq. (0.6) also suggests no difference between a physical bar, with area, and a theoretical elastic line of the same stiffness.

We proceed in the same way for describing beam bending. Before, we said that the beam *axis* becomes curved, but what is this axis? We can select any given horizontal plane within the original (and horizontal) beam for now (which in side view manifests as a line).

Successive planes above and below this must curve differently, in order to preserve their separation when there are no stresses and thus no straining through the depth. Imagine now a uniformly curved portion of beam subtending θ on the reference plane. Another plane, originally y above, has a current length $(y + R)\theta$, where $\kappa = 1/R$ is the reference plane curvature. The plane above becomes strained by an amount $y\kappa$, comparing its lengths before and after curving.

The linear variation of strain with height y leads to the familiar assumption that *plane sections remain plane*, and a given cross-sectional plane rotates during curving (about a line on the horizontal reference plane).

From Hooke's Law, the axial stresses are linear too and produce a turning effect about the reference plane. Their aggregation is another stress resultant, our *bending moment*. Its formal calculation considers an elemental force, $\sigma \cdot w(y)\delta y$, where $w(y)$ is the current width of cross-section at height y; hence $M = \int \sigma w(y) y \mathrm{d}y$

Replacing σ by $E\epsilon$ with $\epsilon = y\kappa$, we find $M = E\kappa \int y^2 w(y)\mathrm{d}y$. The integral term is the *second moment of area* (from the second-degree variation), abbreviated to I, setting $M = EI\kappa$, our generalised Hooke's Law for beam bending: of a stress resultant, its commensurate deformation, and a stiffness term made of a material constant and a geometrical property of the cross-section.

We also have a uni-dimensional expression, analogous to curving of an elastic line with EI as its *bending stiffness*; I, of course, is calculated from the actual cross-section.

To perform its integration, we must know the integration limits for y, which depend on the position of the reference plane. There is no axial force applied, demanding axial equilibrium of the stresses from bending.

We already know each elemental force, and formally integrating them over the cross-section sets $\int \sigma w(y)\mathrm{d}y = 0$: or, $E\kappa \int y w(y)\mathrm{d}y = 0$. Writing $w(y)\mathrm{d}y$ as an elemental area, $\mathrm{d}A$, we therefore observe $\int y\mathrm{d}A = 0$, which is how we locate the *centroid* of the cross-section in the y direction: the reference plane passes through it.

The axial stresses can now be determined from cross-sectional properties. Given that $\epsilon = \sigma/E$ also equals $y\kappa$, with $\kappa = M/EI$, we can re-arrange and obtain $\sigma = My/I$.

Energy Methods

Energy methods ultimately describe the equilibrium or kinematic behaviour of a structure, usually from applying a work 'recipe' for the relevant parameters. For example, the method of *Virtual Work* is a statement of internal energy stored *vs* external effort, where, for a truss:

$$\Sigma_{\text{joints}} W \cdot \Delta = \Sigma_{\text{bars}} T \cdot e. \tag{0.7}$$

The external loads applied to pin-joints are W, and joint displacements are Δ; bar tensions are T and axial extensions, e. Note that W and T are always in equilibrium, and Δ forms a *compatible set* with e in which the joints displace in exact accordance with how the bars extend and rotate.

The energetic terms come from linear elastic behaviour, where normally we expect a 'half' pre-multiplying both sides (even though it would cancel across). This is because Eq. (0.7) is about the effect of *perturbations* – small changes to the configuration of the truss system. For example, if a loaded joint is displaced a little more, the value of applied load does not change.

The Virtual Work performed is therefore small amounts of surplus energy and effort from imposing extra, or *virtual*, loads and displacements *etc.* The operation of Eq. (0.7) also decouples equilibrium from compatibility: our virtual equilibrium set applies to the actual kinematics, as do any virtual kinematics upon the real equilibrium set.

Virtual Work can also be used to provide new theorems[1] such as the *Lower* and *Upper Bound Theorems*, whose purpose guarantees different outcomes for a loaded structure in view of failure.

If we are able to find *any* equilibrium solution that nowhere violates material yielding, the structure will stand safely; but it will collapse if we can postulate a mechanism compatible with how members fail. These seem obvious statements but are non-trivial to prove.

When the structure is originally indeterminate, there is more than one solution for both, with different loading values. They overlap exactly when the values are the same, and a safe structure is about to collapse.

We are not, however, obliged to find the correct solution, which may reduce our work considerably but which may lead to conservative behaviour. Consequently, a Lower Bound solution always gives a safe loading value that can potentially be increased, and an Upper Bound collapse load can potentially reduce.

[1] J Heyman, *Basic Structural Theory*, Appendix B, Cambridge University Press, 2008.