Game Theory Basics

Game theory is the science of interaction. This textbook, derived from courses taught by the author and developed over several years, is a comprehensive, straightforward introduction to the mathematics of non-cooperative games. It teaches what every game theorist should know: the important ideas and results on strategies, game trees, utility theory, imperfect information, and Nash equilibrium. The proofs of such results – in particular, existence of an equilibrium via fixed points and an elegant direct proof of the minimax theorem for zero-sum games – are presented in a self-contained, accessible way. Complementary to that are chapters on combinatorial games such as Go, and – as introductions to algorithmic game theory – traffic games and the geometry of two-player games. This detailed and lively text requires minimal mathematical background and includes many examples, exercises, and pictures. It is suitable for self-study or introductory courses in mathematics, computer science, or economics departments.

Bernhard von Stengel, educated in Germany and the US, is a mathematical game theorist at the London School of Economics and Political Science, and an authority on computational and geometric methods for solving games. He chaired the 2016 World Congress of the Game Theory Society, and is Co-Editor of the *International Journal of Game Theory*.

"This looks like a fine introduction to game theory, *inter alia* emphasizing methods for computing equilibria, and mathematical aspects in general. Especially worthy of note is the chapter devoted to correlated equilibria, a topic of central importance not normally covered in introductory texts."

Robert Aumann, *The Hebrew University of Jerusalem, Nobel Memorial Prize in Economic Sciences 2005*

"This book is a delightful adventure into the mathematics of game theory. Without any heavy apparatus, it lets us into the secrets of a whole range of exciting results that are usually thought too advanced for the common herd. It is not only undergraduate students who will benefit from reading this book. Professional game theorists will find it very useful too."

Ken Binmore, *University College London*

"This is a rather reader-friendly, engaging, and polished superior creation. It illustrates, explains, motivates every definition, theorem, proof. Interesting and unique choice of topics, such as a delightful introductory chapter on combinatorial games. Highly recommended."

Aviezri Fraenkel, *Weizmann Institute of Science, Israel*

"A masterful presentation of mathematical game theory in all its beauty and elegance, from basic notions to advanced techniques. It fills the gaps left by the many textbooks that cover concepts and applications, but devote only the bare minimum to the mathematical tools and insights, without which game theory would not have become the success it is today."

Sergiu Hart, *The Hebrew University of Jerusalem*

"Game theory is the child of mathematicians, as this textbook demonstrates through self-contained, elegant proofs of all seminal theorems. The lively and rigorous exposition of carefully selected models, such as bargaining, combinatorial, and congestion games (the latter two rarely the stuff of textbooks), explains its success far beyond mathematics. To reach deep results on both sides of the theory, Bernhard von Stengel's marvellous learning tool uses uncompromising, yet accessible mathematics, and chooses examples to maximal effect."

Hervé Moulin, *University of Glasgow*

"This book is a gem. The presentation is clear and well structured, often with nice geometric illustrations. It moves step by step from basics to powerful concepts, methods, and results. It is ideal for students of mathematics, computer science, and economics who are curious about what game theory is and how it can be used."

Jörgen Weibull, *Stockholm School of Economics*

"An exceptionally lucid introduction to the fundamentals of game theory, enlivened by examples that are sure to captivate students."

Peyton Young, *University of Oxford*

"This is a rigorous, yet accessible introduction to mathematical non-cooperative game theory. In addition to the coverage of the basic concepts and results, it includes special and advanced topics and applications usually not contained in game theory textbooks, such as combinatorial games, congestion games, and inspection games. The special emphasis on algorithmic and computational techniques makes this textbook, just like its author, a valuable bridge between game theory and computer sciences."

Shmuel Zamir, *The Hebrew University of Jerusalem*

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Preface

This book is an introduction to the mathematics of non-cooperative game theory. Each concept is explained in detail, starting from a main example, with a slow-paced proof of each theorem. The book has been designed and tested for self-study, and as an undergraduate course text with core chapters and optional chapters for different audiences. It has been developed over 15 years for a one-semester course on game theory at the London School of Economics and the distance learning program of the University of London, attended each year by about 200 third-year students in mathematics, economics, management, and other degrees. After studying this book, a student who started from first-year mathematics (the basics of linear algebra, analysis, and probability) will have a solid understanding of the most important concepts and theorems of non-cooperative game theory.

The intended audience are primarily students in mathematics, computer science, and mathematical economics. For mathematicians, we provide complete self-contained proofs (in an economics course, these may be used as reliable background and reference material). For computer scientists, we introduce important ideas in algorithmic game theory, such as traffic equilibria, the "parity argument" for the existence and computation of Nash equilibria, and correlated equilibria. For economists, we use some important economic models as examples, such as Cournot's quantity competition, which is the oldest formal definition of a game. Commitment games are applied to Stackelberg leadership, and in Chapter 11 to the iterated-offers bargaining model. However, given the many good introductions to game theory for economists (for example, Gibbons, 1992, or Osborne, 2004), economic applications are not central to this book.

Aims and Contents

The first aim of this book is to let the student become fluent in game theory. The student will learn the modeling tools (such as game trees and the strategic form) and methods for analyzing games (finding their equilibria). This is provided in the core chapters on non-cooperative games: Chapter 3 on games in strategic form introduces important games such as the Prisoner's Dilemma or Matching Pennies, and the concept of Nash equilibrium (in the book nearly always just called "equilibrium" for brevity). Chapter 4 treats game trees with perfect information. Chapter 6 explains mixed strategies and mixed equilibrium. Game trees with imperfect information are covered in Chapter 10.

A second aim is to provide the conceptual and mathematical foundations of the theory. Chapter 5 explains how the concept of expected utility represents a consistent preference for risky outcomes (and not "risk neutrality" for monetary payoffs, a common confusion). Chapter 7 proves Brouwer's fixed-point theorem, used by Nash (1951) to show the existence of an equilibrium point. Chapter 8 gives a self-contained two-page proof of von Neumann's minimax theorem for zero-sum games. These chapters form independent "modules" that can be used for reference and that can be omitted in a taught course that has less emphasis on mathematical proofs. If only one proof from these chapters is presented, it should be the short proof of the minimax theorem.

A third aim is to introduce ideas that *every game theorist should know* but which are not normally taught, and which make this book special. They are mathematical highlights, typically less known to economists:

- An accessible introduction to combinatorial games in Chapter 1, pioneered by Berlekamp, Conway, and Guy (2001–2004), with a focus on impartial games and the central game of Nim.
- Congestion games, with the famous Braess paradox where increasing network capacity can worsen congestion, and where an equilibrium is found with the help of a potential function (Chapter 2).
- An elegant constructive proof of Sperner's lemma, used to prove Brouwer's fixed-point theorem, and the Freudenthal simplicial subdivision that works naturally in any dimension (Sections 7.4 and 7.7 in Chapter 7).
- The geometry of two-player games (which is my own research specialty) and the algorithm by Lemke and Howson (1964) for finding a Nash equilibrium, which implies that generic games have an odd number of equilibria (Chapter 9).
- An introduction to correlated equilibria in Chapter 12, with an elegant existence proof due to Hart and Schmeidler (1989) that does not require a fixed-point theorem but only von Neumann's minimax theorem.

These chapters are optional. I teach the course in variations but always include Chapter 1 on combinatorial games; they are much enjoyed by students, who would not encounter them in an introductory economics course on game theory. These chapters give also a short introduction to *algorithmic game theory* for computer science students.

The general emphasis of the book is to teach *methods, not philosophy*. Game theorists tend to question and to justify the approaches they take, for example the concept of Nash equilibrium, and the assumed common knowledge of all players about the rules of the game. These questions are of course very important. In fact, these are probably the very issues that require a careful validation in a practical game-theoretic analysis. However, this problem is not remedied by a lengthy discussion of why one should play Nash equilibrium.

I think that a student of game theory should first learn the central models of the strategic and extensive form, and be fluent in analyzing them. That toolbox will then be useful when comparing different game-theoretic models and the solutions that they imply.

Mathematical and Scholarly Level

General mathematical prerequisites are the basic notions of linear algebra (vectors and matrices), probability theory (independence, conditional probability), and analysis (continuity, closed sets). Rather than putting them in a rarely read appendix, important concepts are recalled where needed in text boxes labeled as **Background material**.

For each chapter, the necessary mathematical background and the required previous chapters are listed in a first section on prerequisites and learning objectives.

The mathematics of game theory is not technically difficult, but very conceptual, and requires therefore a certain mathematical maturity. For example, combinatorial games have a recursive structure, for which a generalization of the mathematical induction known for natural numbers is appropriate, called "top-down induction" and explained in Section 1.3.

Game-theoretic concepts have precise and often unfamiliar mathematical definitions. In this book, each main concept is typically explained by means of a detailed introductory example. We use the format of definitions, theorems and proofs in order to be precise and to keep some technical details hidden in proofs. The proofs are detailed and complete and can be studied line by line. The main idea of each proof is conveyed by introductory examples, and geometric arguments are supported by pictures wherever possible.

Great attention is given to details that help avoid unnecessary confusions. For example, a random event with two real values x and y as outcomes will be described with a probability p assigned to the *second* outcome y, so that the interval $[0, 1]$ for p corresponds naturally to the interval $[x, y]$ of the expectation $(1 - p)x + py$ of that event.

This book emphasizes *accessibility, not generality*. I worked hard (and enjoyed) on presenting the most direct and overhead-free proof of each result. In studying these proofs, students may encounter for the first time important ideas from topology, convex analysis, and linear programming, and thus may become interested in the more general mathematical theory of these subjects.

Bibliographic references, historical discussions, and further extensions are deferred to a final section in each chapter, to avoid distractions in the main text. References to further reading may include popular science books that I found relevant and interesting. This book is primarily directed at undergraduate students and not at researchers, but I have tried to be historically accurate. For important works in French or German I have also cited English translations where I could find them.

References to statements inside this book are in upper case, like "Theorem 2.2", and to other works in lower case, like "theorem 13.6 of Roughgarden (2016)".

I use both genders to refer to players in a game, in particular in two-player games to distinguish them more easily. Usage of the pronoun "she" has become natural, and (like "he") it may stand collectively for "he or she".

Each chapter concludes with a set of exercises. They test methods (such as how to find mixed equilibria), or the understanding of mathematical concepts used in proofs (in particular in Chapters 5 and 7). Complete solutions to the exercises are available from the publisher for registered lecturers.

Acknowledgments

Some of the material for this book has been adapted and developed from a course guide originally produced for the distance learning International Programme offered by the University of London (`http://london.ac.uk/`). Rahul Savani and George Zouros gave valuable comments on that first course guide. Many students, some of them anonymously, have pointed out errors and needs for clarification in the subsequent versions of lecture notes. Rahul Savani proof-read the final version with great care. Vanessa Wells drew the nice animal pictures on the cover and in Figures 8.1 and 9.6. I thank Robert Aumann, Paul Dütting, Aviezri Fraenkel, Sergiu Hart, Urban Larsson, Abraham Neyman, Richard Nowakowski, David Tranah, Peyton Young, and Shmuel Zamir for further suggestions.

Martin Antonov is the current lead programmer of the *Game Theory Explorer* software (`http://www.gametheoryexplorer.org`), a joint project with Rahul Savani and (as part of the *Gambit* software) Theodore Turocy, for solving games in strategic and extensive form. From Game Theory Explorer, one can export games into the LaTeX formats used in this book for drawing payoff tables and game trees (available on request).