COARSE GEOMETRY OF TOPOLOGICAL GROUPS

This book provides a general framework for doing geometric group theory for many non-locally compact topological transformation groups that arise in mathematical practice, including homeomorphism and diffeomorphism groups of manifolds, isometry groups of separable metric spaces and automorphism groups of countable structures. Using Roe's framework of coarse structures and spaces, the author defines a natural coarse geometric structure on all topological groups. This structure is accessible to investigation, especially in the case of Polish groups, and often has an explicit description, generalising well-known structures in familiar cases including finitely generated discrete groups, compactly generated locally compact groups and Banach spaces. In most cases, the coarse geometric structure is metrisable and may even be refined to a canonical quasi-metric structure on the group. The book contains many worked examples and sufficient introductory material to be accessible to beginning graduate students. An appendix outlines several open problems in this young and rich theory.

Christian Rosendal is Professor of Mathematics at the University of Maryland. He received a Simons Fellowship in Mathematics in 2012 and is Fellow of the American Mathematical Society.

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Coarse Geometry of Topological Groups

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Preface

The present book is the product of research initiated during an extended sabbatical financed in part by a fellowship by the Simons Foundation. Although the basic discovery of an appropriate coarse structure on Polish groups really materialised around 2013, the initial seeds were sown in several prior works and, in my own case, in the two studies [**77**] and [**78**], where various concepts of boundedness in topological groups were developed.

The main objects of this study are *Polish groups*, that is, separable and completely metrisable topological groups. Although these can of course be treated abstractly, I am principally interested in them as they appear in applications, namely as transformation groups of various mathematical structures and even as the additive groups of separable Banach spaces. This class of groups has received a substantial amount of attention over the past two decades and the results here are my attempt at grappling with possible geometric structures on them. In particular, this is a response to the question of how to apply the language and techniques of geometric group theory, abstract harmonic and functional analysis to their study.

Because this project has been a long time coming, my ideas on the subject have evolved over time and have been influenced by a number of people. Evidently, the work of Mikhail Gromov pervades all of geometric group theory and hence also the ideas presented here. But another specific reference is John Roe's lectures on coarse geometry [**75**] and whose framework of coarse spaces allowed me to extend the definition of a geometric structure to all topological groups and not just the admittedly more interesting subclass of locally bounded Polish groups.

Although this book contains the first formal presentation of the theory, parts of it have already found their way into other publications. In particular, in [**80**] I made a systematic study of equivariant geometry of amenable Polish groups, including separable Banach spaces, and made use of some of the theory presented here. Similarly, in a collaboration with Kathryn Mann [**57**], I investigated the large-scale geometry of homeomorphism groups of compact manifolds within the present framework, while other authors [**19, 38, 99**] have done so for other groups. Finally, [**81**] contains a characterisation of the small-scale geometry of Polish groups and its connection to the large scale.

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