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Introduction

1.1 Motivation

Geometric group theory, or the large-scale geometry of finitely generated discrete groups and of compactly generated locally compact groups, is by now a well-established theory (see [**22, 68**] for recent accounts). In the finitely generated case, the starting point is the elementary observation that the word metrics ρ_Σ on a discrete group Γ given by finite symmetric generating sets $\Sigma \subseteq \Gamma$ are all mutually quasi-isometric, and thus any such metric may be said to define the large-scale geometry of Γ. This has led to a very rich theory weaving together combinatorial group theory, geometry, topology and functional analysis stimulated by the impetus of M. Gromov (see, e.g., [**36**]).

To fix the language, let us recall that a map $(X, d_X) \xrightarrow{\phi} (Y, d_Y)$ between two metric spaces is a *quasi-isometry* provided that there is a constant K so that, for all $x, x' \in X$,

$$\frac{1}{K} d_X(x,x') - K \leqslant d_Y(\phi(x), \phi(x')) \leqslant K \cdot d_X(x,x') + K$$

and also

$$\sup_{y \in Y} \inf_{x \in X} d_Y(y, \phi(x)) \leqslant K.$$

The existence of a quasi-isometry between metric spaces defines an equivalence relation on the class of metric spaces, and hence the large-scale geometry of a finitely generated group Γ is well defined up to this notion of equivalence.

In the locally compact setting, matters have not progressed equally swiftly, even though the basic tools have been available for quite some time. Indeed, by a result of R. Struble [**84**] dating back to 1951, every locally compact second countable group admits a compatible left-invariant *proper* metric, i.e., so that the closed balls are compact. Struble's theorem was based on an earlier well-known result due independently to G. Birkhoff [**11**] and S. Kakutani [**43**] characterising the metrisable topological groups as the first countable topological groups and, moreover, stating that every such group admits a compatible left-invariant metric. However, as is evident from the construction underlying the Birkhoff–Kakutani theorem, if one begins with a compact symmetric generating set Σ for a locally compact second countable group G, then one may obtain a compatible left-invariant metric d that is quasi-isometric to the word metric ρ_Σ induced by Σ. By applying the Baire category theorem and arguing as in the discrete case, one sees that any two such word metrics ρ_{Σ_1} and ρ_{Σ_2} are quasi-isometric, which shows that the compatible left-invariant metric d is uniquely defined up to quasi-isometry by this procedure.

Thus far, there has been no satisfactory general method of studying large-scale geometry of topological groups beyond the locally compact groups, although, of course, certain subclasses such as Banach spaces arrive with a naturally defined geometry. This state of affairs may be largely the result of the presumed absence of canonical generating sets in general topological groups as opposed to the finitely or compactly generated ones. In certain cases, substitute questions have been considered, such as the boundedness or unboundedness of specific metrics [**27**] or of all metrics [**77**]; growth type and distortion of individual elements or subgroups [**34, 72**]; equivariant geometry [**70**] and specific coarse structures [**67**].

In the present book, we offer a solution to this problem that, in many cases, allows one to isolate and compute a canonical word metric on a topological group G and thus to identify a unique quasi-isometry type of G. Moreover, this quasi-isometry type agrees with that obtained in the finitely or compactly generated settings and also verifies the main characteristics encountered there, namely that it is a topological isomorphism invariant of G capturing all possible large-scale behaviour of G. Furthermore, under mild additional assumptions on G, this quasi-isometry type may also be implemented by a compatible left-invariant metric on the group.

Although applicable to all topological groups, our main interest is in the class of *Polish groups*, i.e., separable completely metrisable topological groups. These include most interesting topological transformation groups, e.g.,

$$\mathrm{Homeo}(\mathcal{M}), \qquad \mathrm{Diff}^k(\mathcal{M}),$$

for $\mathcal{M}$ a compact (smooth) manifold, and

$$\mathrm{Aut}(\mathbf{A}),$$

for a countable discrete structure **A**, along with all separable Banach spaces and locally compact second countable groups. Another class that has recently received much attention by geometric topologists is the mapping class groups of *infinite-type* surfaces; that is, so that the mapping class group is not finitely generated. In this case, the mapping class group can be viewed as the automorphism group of an associated countable graph and thus falls into the framework of automorphism groups of countable discrete structures. However, it should be stressed that the majority of our results are directly applicable in the greater generality of *European* groups, i.e., Baire topological groups, countably generated over every identity neighbourhood. This includes, for example, all σ-compact locally compact Hausdorff groups and all (potentially non-separable) Banach spaces.

One central technical tool is the notion of coarse structure due to J. Roe [**74, 75**], which may be viewed as the large-scale counterpart to uniform spaces. Indeed, given an écart (also known as pre- or pseudo-metric) d on a group G, let $\mathcal{E}_d$ be the coarse structure on G generated by the entourages

$$E_\alpha = \{(x, y) \in G \times G \mid d(x, y) < \alpha\},$$

for $\alpha < \infty$. That is, $\mathcal{E}_d$ is the ideal of subsets of $G \times G$ generated by the E_α. In analogy with A. Weil's result [**94**] that the left-uniform structure $\mathcal{U}_L$ on a topological group G can be written as the union

$$\mathcal{U}_L = \bigcup_d \mathcal{U}_d$$

of the uniform structures $\mathcal{U}_d$ induced by the family of continuous left-invariant écarts d on G, we define the *left-coarse structure* $\mathcal{E}_L$ on G by

$$\mathcal{E}_L = \bigcap_d \mathcal{E}_d.$$

This definition equips every topological group with a left-invariant coarse structure, which, like a uniformity, may or may not be metrisable, i.e., be the coarse structure associated to a metric on the group. To explain when that happens, we say that a subset $A \subseteq G$ is *coarsely bounded in* G if A has finite diameter with respect to every continuous left-invariant écart on G. This may be viewed as an appropriate notion of 'geometric compactness' in topological groups and, in the case of a Polish group G, has the following combinatorial reformulation. Namely, $A \subseteq G$ is coarsely bounded in G if, for every identity neighbourhood V, there is a finite set $F \subseteq G$ and a k so that $A \subseteq (FV)^k$.

Theorem 1.1 *The following conditions are equivalent for a Polish group G:*

(1) *the left-coarse structure $\mathcal{E}_L$ is metrisable;*
(2) *G is* locally bounded*, i.e., has a coarsely bounded identity neighbourhood;*
(3) *$\mathcal{E}_L$ is generated by a compatible left-invariant metric d, i.e., $\mathcal{E}_L = \mathcal{E}_d$;*
(4) *a sequence (g_n) eventually leaves every coarsely bounded set in G if and only if there is some compatible left-invariant metric d on G for which $d(g_n, 1) \xrightarrow[n]{} \infty$.*

In analogy with proper metrics on locally compact groups, the metrics appearing in condition (3) above are said to be *coarsely proper*. Indeed, these are exactly the compatible left-invariant metrics all of whose bounded sets are coarsely bounded. Moreover, by Struble's result, on a locally compact second countable group these are the proper metrics.

The category of coarse spaces may best be understood by its morphisms, namely, the bornologous maps. In the case where $(X, d_X) \xrightarrow{\phi} (Y, d_Y)$ is a map between pseudo-metric spaces, then ϕ is *bornologous* if there is an increasing modulus $\theta \colon \mathbb{R}_+ \to \mathbb{R}_+$ so that, for all $x, x' \in X$,

$$d_Y\big(\phi(x), \phi(x')\big) \leqslant \theta\big(d_X(x, x')\big).$$

By using this, we may quasi-order the continuous left-invariant écarts on G by setting $\partial \lll d$ if the identity map $(G, d) \to (G, \partial)$ is bornologous. One then shows that a metric is coarsely proper when it is the maximum element of this ordering. Although seemingly most familiar groups are locally bounded, counterexamples exist, such as the infinite direct product of countably infinite groups, e.g., $\mathbb{Z}^{\mathbb{N}}$.

However, just as the word metric on a finitely generated group is well defined up to quasi-isometry, we may obtain a similar canonicity provided that the group G is actually *generated by a coarsely bounded* set A, that is, every element of G can be written as a product of elements of $A \cup A^{-1} \cup \{1\}$. In order to do this, we refine the quasi-ordering $\lll$ on continuous left-invariant écarts on G above by letting $\partial \ll d$ if there is a constant K so that $\partial \leqslant K \cdot d + K$. Again, if d is maximum in this ordering, we say that d is *maximal*. Obviously, two maximal écarts are quasi-isometric, whence these induce a canonical *quasi-isometry type* on G. Moreover, as it turns out, the maximal écarts are exactly those that are quasi-isometric to the word metric

$$\rho_\Sigma(x,y) = \min(k \mid \exists z_1,\dots,z_k \in \Sigma\colon x = yz_1\cdots z_k)$$

given by a coarsely bounded generating set $\Sigma \subseteq G$.

Theorem 1.2 *The following are equivalent for a Polish group G:*

(1) *G admits a compatible left-invariant maximal metric;*
(2) *G is generated by a coarsely bounded set;*
(3) *G is locally bounded and not the union of a countable chain of proper open subgroups.*

A reassuring fact about our definition of coarse structure and quasi-isometry type is that it is a conservative extension of the existing theory. Namely, as the coarsely bounded sets in a σ-compact locally compact group coincide with the relatively compact sets, one sees that our definition of the quasi-isometry type of a compactly generated locally compact group coincides with the classical definition given in terms of word metrics for compact generating sets. The same argument applies to the category of finitely generated groups when these are viewed as discrete topological groups. Moreover, as will be shown, if $(X,\|\cdot\|)$ is a Banach space, then the norm metric will be maximal on the underlying additive group $(X,+)$, whereby $(X,+)$ will have a well-defined quasi-isometry type, namely, that of $(X,\|\cdot\|)$. But even in the case of homeomorphism groups of compact manifolds M, as shown in [**57, 62**], the maximal metric on the group $\mathrm{Homeo}_0(M)$ of isotopically trivial homeomorphisms of M is quasi-isometric to the fragmentation metric originating in the work of R. D. Edwards and R. C. Kirby [**26**].

1.2 A Word on the Terminology

Some of the basic results presented here have previously been included in the pre-print [**79**], which now is fully superseded by this book. Under the impetus of T. Tsankov, we have changed the terminology from [**79**] to become less specific and more in line with the general language of geometric group theory. Thus, the coarsely bounded sets were originally called *relatively (OB) sets* to keep in line with the terminology from [**77**]. Similarly, locally bounded groups were denoted *locally (OB)* and groups generated by coarsely bounded sets were called *(OB) generated*. For this reason, other papers based on [**79**], such as [**19**], [**57**], [**80**] and [**98**], also use the language of relatively (OB) sets. The translation between the two is straightforward and involves no change in theory.

1.3 Summary of Findings

To aid the reader in the navigation of the new concepts appearing here, we include Figure 1.1, depicting the main classes of Polish groups and a few simple representative examples from some of these. Observe that in the diagram the classes increase going up and from left to right.

Note that the shaded areas reflect the fact that every Polish group of bounded geometry is automatically locally bounded, and that coarsely bounded groups trivially have bounded geometry.

1.3.1 Coarse Structure and Metrisability

Chapter 2 introduces the basic machinery of coarse structures with its associated morphisms of bornologous maps and analyses these in the setting of topological groups. We introduce the canonical left-invariant coarse structure

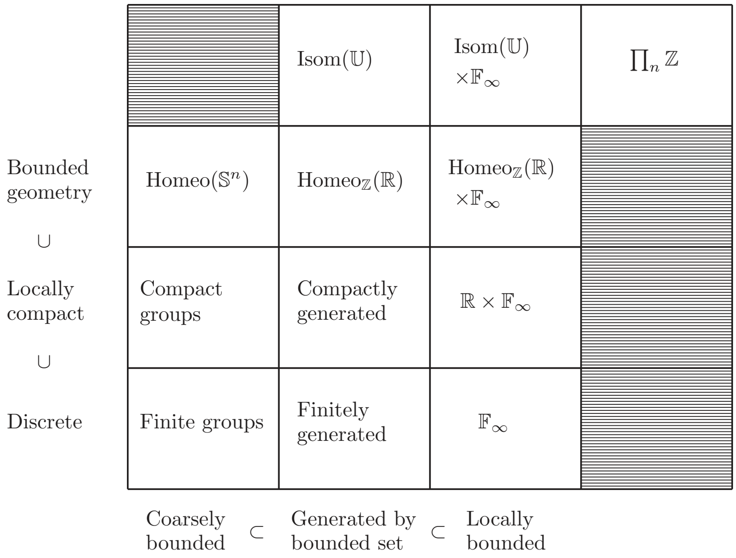

Figure 1.1 Main classes of Polish groups organised according to their coarse geometry.

$\mathcal{E}_L$ with its ideal of coarsely bounded sets and compare this with other coarse structures such as the group-compact coarse structure $\mathcal{E}_{\mathcal{K}}$.

The main results of the chapter concern the identification of coarsely proper and maximal metrics along with Theorems 1.1 and 1.2 characterising the existence of these. This also leads to a version of the Milnor–Schwarz Lemma [**64, 83**] adapted to our setting; this is the central tool in the computation of actual quasi-isometry types of groups.

1.3.2 Basic Structure Theory

In Chapter 3 we provide some of the basic tools for the geometric study of Polish groups and present a number of computations of the geometry of specific groups. The simplest class to consider is that of the 'metrically compact' groups, i.e., those quasi-isometric to a one-point space. These are exactly those coarsely bounded in themselves. This class of groups was extensively studied in [**77**] and includes a large number of topological transformation groups of highly homogeneous mathematical structures such as homeomorphism groups of spheres and the unitary group of separable infinite-dimensional Hilbert space.

The locally Roelcke pre-compact groups comprise another particularly interesting class. This class includes examples such as the automorphism group of the countably regular tree $\mathrm{Aut}(\mathsf{T}_\infty)$ and the isometry group of the Urysohn metric space $\mathrm{Isom}(\mathbb{U})$ that turn out to be quasi-isometric to the tree T_∞ and the Urysohn space $\mathbb{U}$, respectively. Because, by a recent result of J. Zielinski [**99**], the locally Roelcke pre-compact groups have locally compact Roelcke completions, they also provide us with an important tool for the analysis of Polish groups of bounded geometry in Chapter 5.

Indeed, a closed subgroup H of a Polish group G is said to be *coarsely embedded* if the inclusion map is a coarse embedding, or equivalently, a subset $A \subseteq H$ is coarsely bounded in H if and only if it is coarsely bounded in G. Because, in a locally compact group, the coarsely bounded sets are simply the relatively compact sets, every closed subgroup is coarsely embedded, although not necessarily quasi-isometrically embedded in the compactly generated case. However, this fails dramatically for Polish groups. Indeed, every Polish group is isomorphic to a closed subgroup of the coarsely bounded group $\mathrm{Homeo}([0,1]^{\mathbb{N}})$. So this subgroup is coarsely embedded only if coarsely bounded itself. This difference, along with the potential non-metrisability of the coarse structure, accounts for a great deal of the additional difficulties arising when investigating general Polish groups.

Theorem 1.3 *Every locally bounded Polish group G is isomorphic to a coarsely embedded closed subgroup of the locally Roelcke pre-compact group* $\mathrm{Isom}(\mathbb{U})$.

Via this embedding, every locally bounded Polish group can be seen to act continuously on a locally compact space preserving its geometric structure.

The main structural theory of Chapter 3 is a byproduct of the analysis of the coarse geometry of product groups. Indeed, we show that a subset A of a product $\prod_i G_i$ is coarsely bounded if and only if each projection $\mathsf{proj}_i(A)$ is coarsely bounded in G_i. From this, we obtain a universal representation of all Polish groups.

Theorem 1.4 *Every Polish group G is isomorphic to a coarsely embedded closed subgroup of the countable product* $\prod_n \mathrm{Isom}(\mathbb{U})$.

This can be viewed as providing a product resolution of the coarse structure on non-locally bounded Polish groups.

1.3.3 Coarse Geometry of Group Extensions

In Chapter 4 we address the fundamental and familiar problem of determining the coarse geometry of a group G from those of a closed normal subgroup K and the quotient group G/K. That is, we will reconstruct the coarse geometry of the middle term G from those of K and G/K in the short exact sequence

$$1 \longrightarrow K \longrightarrow G \longrightarrow G/K \longrightarrow 1.$$

Although certain things can be said about the general situation, we mainly focus on a more restrictive setting, which includes that of central extensions. Namely, we suppose that K is a closed normal subgroup of a Polish group G, where the latter is generated by K and the centraliser $C_G(K) = \{g \in G \mid \forall k \in K\ gk = kg\}$ of K in G, i.e., such that $G = K \cdot C_G(K)$. Note that, in this case, we also have that

$$G/K = C_G(K)/Z(K),$$

where $Z(K) = \{h \in K \mid \forall k \in K\ hk = kh\}$ is the centre of K. Assume furthermore that K is coarsely embedded in G and that $G/K \xrightarrow{\phi} C_G(K)$ is a section for the quotient map that is bornologous as a map $G/K \xrightarrow{\phi} G$. Then the map

$$K \times G/K \to G, \qquad (k,h) \mapsto k\phi(h)$$

defines a coarse equivalence between $K \times G/K$ and G.

A common instance of this setup is seen when G is generated by a discrete normal subgroup $K = \Gamma$ and a connected closed subgroup F.

Theorem 1.5 *Suppose G is a Polish group generated by a discrete normal subgroup Γ and a connected closed subgroup F. Assume also that $\Gamma \cap F$ is coarsely embedded in F and that $G/\Gamma \xrightarrow{\phi} F$ is a bornologous section for the quotient map. Then G is coarsely equivalent to $G/\Gamma \times \Gamma$.*

In connection with these problems, several fundamental issues emerge:

- When is K coarsely embedded in G?
- When does the quotient map $G \xrightarrow{\pi} G/K$ admit a bornologous section $G/K \xrightarrow{\phi} G$?
- Is G locally bounded provided that K and G/K are?

Indeed, to determine whether K is coarsely embedded in G or whether a section $G/K \xrightarrow{\phi} G$ is bornologous both require some advance knowledge of the coarse structure on G itself. However, the latter is exactly what we are trying to determine. To circumvent this conumdrum, we study the cocycle associated with a section ϕ. Indeed, given a section $G/K \xrightarrow{\phi} C_G(K)$ for the quotient map, one obtains an associated cocycle $G/K \times G/K \xrightarrow{\omega_\phi} Z(K)$ by the formula

$$\omega_\phi(h_1,h_2) = \phi(h_1h_2)^{-1}\phi(h_1)\phi(h_2).$$

Assuming that ϕ is Borel and G/K locally bounded, the coarse qualities of the map $G/K \xrightarrow{\phi} G$ and whether K is coarsely embedded in G now become intimately tied to the coarse qualities of ω_ϕ. Let us state this for the case of central extensions.

Theorem 1.6 *Suppose K is a closed central subgroup of a Polish group G so that G/K is locally bounded and that $G/K \xrightarrow{\phi} G$ is a Borel measurable section of the quotient map. Assume also that, for every coarsely bounded set $B \subseteq G/K$, the image*

$$\omega_\phi\big[G/K \times B\big]$$

is coarsely bounded in K. Then G is coarsely equivalent to $K \times G/K$.

The main feature here is, of course, that the assumptions make no reference to the coarse structure of G, only to those of K and G/K.

We then apply our analysis to covering maps of manifolds or more general locally compact spaces, which builds on a specific subcase from our joint work with K. Mann [**57**]. Our initial setup is a proper, free and cocompact action

$$\Gamma \curvearrowright X$$

of a finitely generated group Γ on a path-connected, locally path-connected and semi-locally simply connected, locally compact metrisable space X. Then the normaliser $N_{\mathrm{Homeo}(X)}(\Gamma)$ of Γ in the homeomorphism group $\mathrm{Homeo}(X)$ is the group of all lifts of homeomorphisms of $M = X/\Gamma$ to X, whereas the centraliser $C_{\mathrm{Homeo}(X)}(\Gamma)$ is an open subgroup of $N_{\mathrm{Homeo}(X)}(\Gamma)$. Let

$$N_{\mathrm{Homeo}(X)}(\Gamma) \xrightarrow{\pi} \mathrm{Homeo}(M)$$

be the corresponding quotient map and let

$$Q_0 = \pi\big[C_{\mathrm{Homeo}(X)}(\Gamma)\big]$$

be the subgroup of $\mathrm{Homeo}(M)$ consisting of homeomorphisms admitting lifts in $C_{\mathrm{Homeo}(X)}(\Gamma)$. We show that Q_0 is open in $\mathrm{Homeo}(M)$. Also, assume H is a subgroup of Q_0 that is Polish in a finer group topology, say H is the transformation group of some additional structure on M, e.g., a diffeomorphism or symplectic group. Then the group of lifts $G = \pi^{-1}(H) \leqslant N_{\mathrm{Homeo}(X)}(\Gamma)$ carries a canonical lifted Polish group topology and is related to H via the exact sequence

$$1 \to \Gamma \to G \xrightarrow{\pi} H \to 1.$$

By using only assumptions on the structure of Γ, we can relate the geometry of G to those of H and Γ.

Theorem 1.7 *Suppose $\Gamma/Z(\Gamma) \xrightarrow{\psi} \Gamma$ is a bornologous section for the quotient map, that $H \leqslant Q_0$ is Polish in some finer group topology and that $G = \pi^{-1}(H)$. Then G is coarsely equivalent to $H \times \Gamma$.*

Observe here that ψ is a section for the quotient map from the discrete group Γ to its quotient by the centre, which a priori has little to do with H and G. Nevertheless, a main feature of the proof is the existence of a bornologous section $H \xrightarrow{\phi} C_G(\Gamma)$ for the quotient map π, which is extracted from ψ.

Also, applying our result to the universal cover $X = \tilde{M}$ of a compact manifold M, we arrive at the following result.