

## Rigid Body Dynamics

Building up from first principles and simple scenarios, this comprehensive introduction to rigid body dynamics gradually introduces readers to tools to address involved real-world problems, and cutting-edge research topics. Using a unique blend of conceptual, theoretical, and practical approaches, concepts are developed and rigorously applied to practical examples in a consistent and understandable way. It includes discussion of real-world applications including robotics and vehicle dynamics, and over 40 thought-provoking fully worked examples to cement readers' understanding.

Providing a wealth of resources allowing readers to confidently self-assess – including over 100 problems with solutions, over 400 high-quality multiple choice questions, and end-of-chapter puzzles dealing with everyday situations – this is an ideal companion for undergraduate students in aerospace, civil, and mechanical engineering.

**Joaquim A. Batlle** is Emeritus Professor of Mechanical Engineering at the Universitat Politècnica de Catalunya, and a member of the Royal Academy of Sciences and Arts of Barcelona.

**Ana Barjau Condomines** is Associate Professor of Mechanical Engineering at the Universitat Politècnica de Catalunya.

“As a teacher of mechanics to engineering graduate students for more than 30 years, I have always been inspired by the particular clear and rigorous style followed by the authors to present the concepts and ideas of mechanics. This second volume in this series on mechanics of particles and rigid bodies culminates their work in collecting a lifelong experience into this book series. The explicit notation and language and the numerous examples and exercises make it a perfect learning companion. This book on dynamics, along with its kinematics companion, are my first choices as teaching reference books.”

Javier Ros, Public University of Navarre (UPNA)

“A rigorous and thorough account of the dynamics of systems of rigid bodies, this book is an invaluable reference for the practitioner and a foundational teaching guide for the student. A treasure trove of examples, exercises and quizzes provides the necessary additional scaffolding for a real understanding of the subject.”

Juan Reyero, [greaterskies.com](http://greaterskies.com)

“*Rigid Body Dynamics* is a textbook of classical mechanics highly recommended for engineering students, because the principles of rigid body dynamics are applied to practical cases. The authors present a textbook following the style of their previous work (*Rigid Body Kinematics*), of which this textbook is a natural continuation. The textbook is beautifully illustrated with realistic schemes that make a clear and easily understandable text. . .”

Maria Rosario Isabel Lopez Hermoso, University of Barcelona

“This comprehensive introduction to rigid body dynamics is a tour de force. The information contained in it is clearly the outcome of distilling a lot of information over years, to come out with an outstandingly clear text with a connecting thread so well designed that no gap is left in the discussion. Every single point is perfectly motivated and proved. The beautifully illustrated examples and exercises are an excellent complement to the theoretical part. As a result, this book is ideal not only for the classroom but also for self-study. From start to finish, it is an instant classic and, from my point of view, one of the very best on the topic.”

Federico Thomas, Spanish National Research Council (CSIC)

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Joaquim A. Batlle , Ana Barjau Condomines  
Frontmatter  
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JOAQUIM A. BATLLE

Universitat Politècnica de Catalunya, Barcelona

ANA BARJAU CONDOMINES

Universitat Politècnica de Catalunya, Barcelona



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## Preface

### Origin and Scope

*Rigid Body Dynamics* provides all the tools necessary to solve the dynamics of systems of rigid bodies in three dimensions. Its contents cover the syllabus of the last two-thirds of a year-long course for advanced undergraduate students. The first third of such a course corresponds to rigid body kinematics and receives an exhaustive treatment in our other text, *Rigid Body Kinematics*, available from the same publisher. We have taught this course at the School of Industrial Engineering of the Universitat Politècnica de Catalunya (UPC) for many years.

There are many good texts on dynamics. However, after more than 40 cumulated years of teaching a subject, many a lecturer will tend to develop a view of how the subject taught can be best presented. Writing a textbook may be seen as the last step in honing that view. Our book was meant not just to produce a text fully covering a syllabus as taught, but also to uphold both rigor and generality. In the end we wish to present our readers with a tool that allows them to face any rigid body dynamics problems they might encounter in different fields of engineering, such as robotics, automation, and biomechanics.

Though initially intended for an undergraduate course, the book may be used in courses for graduate students, or as a reference for practitioners and even researchers.

### Prerequisites

The reader should be acquainted with algebra and calculus involving scalar functions and vectors, and have a solid background in kinematics.

### Main Features

This textbook deals basically with the dynamics of multibody systems moving in three dimensions. Planar dynamics and statics appear as particular applications of that general case. The main focus is on the vector approach of mechanics, though we also include an introduction to analytical mechanics (Lagrange's equations).

The notation deserves a special comment. We wanted our notation to be both explicit and self-sufficient (as the IUPAC chemical nomenclature!), meaning that the details needed to fully understand every term in an equation would always be present. For that reason, we opted for a very precise notation that allows readers to know exactly “where they are.” In particular, we systematically include the reference frame as a subscript in all variables that are frame-dependent. In classical mechanics, that information is of paramount importance, as the laws of dynamics are not written in the same way in inertial and noninertial frames.

Our long teaching experience tells us that students like to skip proofs and simply practice with the equations on application examples. Notwithstanding this, one needs to fully understand every term in an equation in order to apply it successfully. This is why we organize the presentation of every new equation (method or theorem) according to the same sequence, namely we:

- define accurately each term that appears in the theorem (or method) so that the reader is able to calculate it;
- write down the theorem;
- give the proof after the equation (theorem or method) is written, isolated from the text (so it is both easy to skip it and to come back to it later).

Advanced topics are presented in appendices. They may be overlooked without impairing comprehension of the following chapters.

There is hardly a page without a figure: they help understand and visualize concepts, theorems, and their properties. Every theorem or method is illustrated by many solved examples, and a long list of multiple-choice quizzes and application exercises (both with answers or results). Finally, puzzles are proposed at the end of most chapters. They reflect realistic everyday situations and are fully solved and commented on. Each one includes at least one figure.

## Content Description

The book begins with a presentation of vector dynamics, starting with Newton’s laws of particle dynamics. We focus on vector theorems, complemented by the method of virtual power and the work–energy theorem. For the sake of generality, all theorems are formulated both in inertial and noninertial reference frames.

Chapter 1 is devoted to the principles of Newtonian mechanics, which govern the dynamics of particles. Unlike most textbooks in mechanics, we present Mach’s “empirical propositions” as an alternative to Newton’s laws, and discuss the equivalences between both approaches. The most usual interactions between particles are then described and formulated (when possible).

Chapter 2 describes the interactions between rigid bodies with a particular emphasis on constraint forces. It includes the rigorous treatment of links with total and tangent redundancy, and proposes a systematic tool to analyze them. This formulation is original and is not found in any other textbooks dealing with this subject.

Chapter 3 introduces all concepts related to the mass distribution of the system (center of mass and inertia tensor about a point) needed to apply the vector theorems that solve the dynamics of a rigid body. We provide a few theorems that help to calculate those elements, and analyze the consequences of the mass distribution geometry on them.

Chapter 4 constitutes the core of this textbook. It introduces the vector theorems for the case of systems with constant matter: the linear momentum theorem (LMT) and the angular momentum theorem (AMT). Both relate the rate of change of a vector associated with the system (the linear momentum and the angular momentum about a point, respectively) to the net interactions (net force in the LMT, net moment in the AMT) exerted on the system. We pay particular attention to rotational dynamics, where the behavior of rigid bodies is often counterintuitive. The dynamic role of the principal axes of inertia is discussed through several didactic examples.

Chapter 5 is devoted to the work–energy theorem. This theorem plays an important role in physics: it is the bridge linking mechanics to many other branches (such as electromagnetism and thermodynamics) as it deals with energy – a magnitude defined and used in all of them.

As a first step toward analytical dynamics, Chapter 6 presents the method of virtual power, central and instrumental in many areas of the theory of machines and mechanisms, particularly in 2D (planar) situations. Its analytically systematized version leads us to the Lagrange equations, which are presented in Chapter 7. They are specially designed to find equations of motion, information indispensable in automation and robotics. They are also the gateway to analytical dynamics.

Finally, Chapter 8 concludes the book with an introduction to percussive dynamics that presents the percussive version of the procedures of dynamics: vector theorems, the energy theorem, the virtual power method, and the Lagrange equations. Newton's restitution rule, used extensively in the scientific literature but not always in a consistent way, is revisited with a rigor unusual in many texts on classical mechanics.

## Acknowledgments

A textbook is never a project disconnected from its academic context. Rather it is the result of countless interactions between the authors and others, involved in one way or another in their lectures on the subject. Members of the teaching team and students over the past 30 years have had a positive influence on the way we present the subject today. Our thanks go to all of them.

We are also grateful for the detailed work done by a few exceptional reviewers, who went through the draft versions of this book. Their comments and suggestions have been precious to us.

Last but not least, we would like to extend our gratitude to two Cambridge University Press publishers, who consistently provided us with first-class assistance throughout the elaborate process of putting this text together. In that, Julia Ford, Stephen Elliot, Jenny van der Meijden and Niranjana Harikrishnan deserve a very special mention.

## Abbreviations

AMT	angular momentum theorem
CAE	constraint auxiliary element
CAI	central axis of inertia
CoR	coefficient of restitution
DoIC	degree of ill-conditioning
DoF	degree of freedom
ICR	instantaneous center of rotation
ISA	instantaneous screw axis
LMT	linear momentum theorem
MVP	method of virtual power
PAI	principal axis of inertia
PC	percussion center
RBK	rigid body kinematics
WET	work–energy theorem