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978-1-108-84213-6 — Rigid Body Dynamics
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# 1 Particle Dynamics

Particle dynamics is the starting point for the vector formulation of dynamics. It was first formulated by Isaac Newton through three fundamental laws (***Newton's laws of motion***), which are the core of the *Philosophiæ Naturalis Principia Mathematica* (Isaac Newton's major work). These laws have been reformulated many times since they were published, not only to adapt the language to a more modern and understandable form (and that includes using mathematical formulations not existing in Newton's times) but also to overcome several axiomatic issues as "are those laws independent? (is the first law a particular case of the second law?)," "are they laws or definitions? (is the second one a law or just a definition of force?)." Though these laws are widely known and used, their deep understanding is not straightforward. Concepts such as mass and force, which are at the core of that formulation, are not at all simple, and have been criticized by many scientists over the centuries.

Ernst Mach's axiomatization of Newtonian mechanics (presented in his book *The Science of Mechanics*) is among the most important reformulations, and it is the one we will present in this chapter. It does not mean that it is the "final" formulation: it has also been subjected to criticism by other scientists, and reformulated in turn!

Mach's new axiomatization is based on three **empirical propositions**. We will show that, together with the consideration of the properties of space and time and a precise definition of the concepts mass and force, those principles lead to ***Newton's laws of motion***. Once this is proved, we will stick to Newton's formulation.

The fundamental law governing particle dynamics (Newton's second law) is presented both in Galilean and non-Galilean reference frames. A discussion on the frames which appear to behave as Galilean ones (according to the scope of the problem under study) is also included.

Finally, we present the most usual interactions acting on particles and provide a formulation for the gravitational attraction, forces associated with springs and dampers, friction, and a description of constraint forces.

## 1.1 Fundamental Assumptions Underlying Newtonian Dynamics

Newtonian dynamics rely on a few definitions and laws (assumptions that cannot be proved) from which many useful theorems may be deduced. There is not a unique way of formulating those laws. Though Newton was the first to lay the framework for classical mechanics, other scientists have proposed different systems of axioms and definitions to overcome the drawbacks of Newton's formulation.

Whatever the formulation is, there are, however, two principles which are assumed in Newtonian mechanics, and which deserve some comments: the ***principle of causality*** and the ***principle of absolute simultaneity***.

## Principle of Causality

Dynamics studies the motion of material objects as a function of the physical factors affecting them. In the Newtonian formulation, those physical factors are mainly their *mass* and the *forces* acting on the objects. Whatever the forces may be, all definitions assume implicitly that there is a correlation between them and the object's motion. This correlation is sometimes described in a simplistic way as a causality relationship: "Dynamics is the study of the motion of bodies *caused by* the action of forces."

The equations of Newtonian dynamics are differential equations relating the second order derivative of the objects generalized coordinates to the coordinates and their first derivatives, all of them at a same time instant. This simultaneity of motion variables makes it difficult (and formally impossible) to distinguish "causes" and "effects," as the former should actually precede the latter. That distinction becomes a convention.

## Principle of Absolute Simultaneity

The ***principle of absolute simultaneity***[1] states, in short, that a sequence (order of succession) of events is the same for all observers (or independent from the reference frame). In Newton's conception of the universe, **absolute time** goes hand in hand with **absolute space**: they constitute an immutable stage where physical events occur, they are independent external realities.

## Other Assumptions

Last but not least: as in any scientific theory, Newtonian dynamics have a limited field of application (directly related to the experimental limitations of the seventeenth century!). Outside that field, it yields inaccurate results. The main limitations are:

- Low-speed dynamics: objects moving with speeds comparable to the speed of light cannot be treated successfully with this theory.
- Medium length scale dynamics: molecular dynamics and long-reach astronomy are also out of scope.
- Electromagnetic phenomena are excluded.

[1] Batlle, J.A. and Barjau Condomines, A. (2019) *Rigid Body Kinematics*, Cambridge University Press, chapter 1.

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## 1.2 Galilean and Non-Galilean Reference Frames

When we enter the field of Newtonian dynamics, we are confronted with a surprising fact: all reference frames are not equivalent for the formulation of the dynamical equations (or of the fundamental laws of Newtonian dynamics).

The methods presented in kinematics (composition of movements, rigid body kinematics. . .) apply equally in all reference frames.

The situation in dynamics is quite different: as dynamics intends to relate movement and "causes," as the movement depends in principle on the reference frame, the "causes" may vary from one frame to another.

What can be the "cause" of motion (or motion change) of a particle?[2] Certainly, the existence of other material objects, which may interact with the particle (that is, have an effect on its motion). But the observation of reality suggests other factors that might have a consequence on the particle motion. Given a frame where the observations or experiments are performed, those factors are:

- the particular position of the interacting system in the reference frame;
- the particular orientation of the interacting system in the reference frame;
- the particular time instant of the observations.

The following example is an illustration of those three possibilities. We assume that the reader is acquainted with some very basic aspects of particle dynamics (to be presented later on in this chapter).

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► **Example 1.1** Let's consider the following situations:

(a) Particle **P** is initially at rest on a small smooth horizontal surface (whose area is much smaller than that of the Earth) close to the Earth's surface (Fig. 1.1a). If we consider only the horizontal motion (thus reducing the problem from 3D to 2D), the absence of roughness is equivalent to zero horizontal interaction, and **P** behaves as a free particle.

(b) Particle **P** is initially launched with a given speed in any horizontal direction and from any location on a smooth horizontal surface close to the Earth's surface (Fig. 1.1b). Under these circumstances, **P** is again a free particle.

(c) Particle **P** is in a smooth slot in a support and attached to two identical springs with ends fixed to that support (Fig. 1.1c). If we consider only the rectilinear motion in the slot, the particle and the springs constitute the interacting system. The support will be glued anywhere and with any orientation on a horizontal surface close to the Earth's surface.

[2] Newton did not formulate the laws for "particles" but for "bodies." However, updated formulations of those laws talk of "particles" (short denomination for a mass-point object). Further on, we will present the concept "particle" as the simplest model for a material object without any considerations on its size, as far as its orientation is irrelevant for the situation under study.

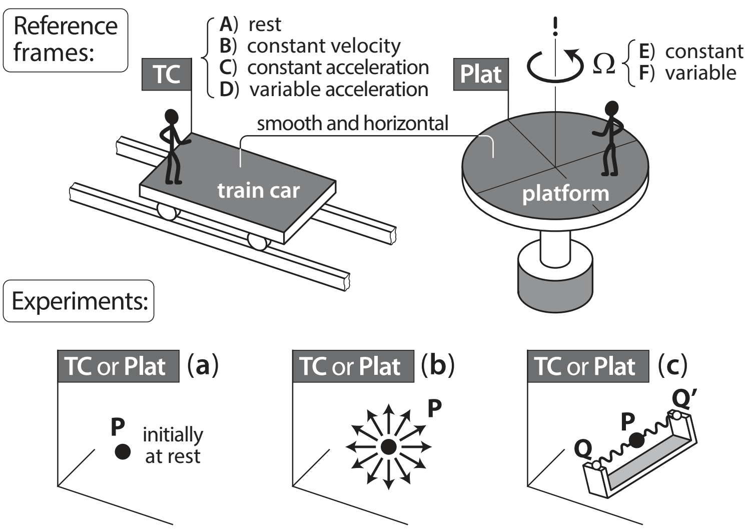


Fig. 1.1

In situations (1) and (2), we want to observe the evolution of the initial state (initial position and velocity of the particle). In situation (3) we will be interested in the equilibrium position of the particle.

The reference frame will be that of the smooth surface (which coincides with the springs support in the third situation). Let's consider different choices:

A. Reference frame at rest with respect to the Earth.
B. Reference frame with a uniform rectilinear motion with respect to the Earth.
C. Reference frame with a rectilinear motion with constant acceleration with respect to the Earth.
D. Reference frame with a rectilinear motion with variable acceleration with respect to the Earth.
E. Reference frame rotating with constant angular velocity about a fixed axis with respect to the Earth.
F. Reference frame rotating with variable angular velocity about a fixed axis with respect to the Earth.

The reference frames **A** to **D** can be pictured as a train car with different motions on a horizontal straight railway, while the reference frames **E** and **F** can be assimilated to a platform rotating about an Earth-fixed axis.

Train cars and rotating platforms are observation reference frames familiar enough to the reader, so that the result of those three thought experiments can be correctly guessed. For instance, leaving the particle **P** at rest on a train car in different positions will have no consequences, whereas the time instant when this is done does have an influence on

**Table 1.1**

| Reference frame | Position dependence | Orientation dependence | Time dependence |
|---|---|---|---|
| A | NO | NO | NO |
| B | NO | NO | NO |
| C | NO | **YES** | NO |
| D | NO | **YES** | **YES** |
| E | **YES** | **YES** | NO |
| F | **YES** | **YES** | **YES** |

the observations when the train car has a variable velocity relative to the Earth. However, when leaving **P** at rest on a rotating platform, different initial positions yield different results: for instance, if located just on the platform center, the particle stays at rest.

Table 1.1 summarizes the influence of position, orientation and time instant for experiments (1), (2), and (3) in reference frames **A** to **F**. A "YES" means that at least one of the experiments does show a dependence on position/orientation/time instant. ◀

Example 1.1 suggests that there are reference frames (**A** and **B**) which do not have any influence on the motion of particles because, *as far as mechanical phenomena are concerned*:

- All positions are equivalent: space is **homogeneous**.
- All orientations are equivalent: space is **isotropic**.
- All time instants are equivalent: time is **uniform**.

Those reference are called **inertial** or **Galilean reference frames**.

Example 1.1 also suggests the existence of reference frames (**C** to **F**) where some of those properties are not fulfilled: those are non-inertial or non-Galilean reference frames.

The formulation of the dynamics of a mechanical system will be simpler in a Galilean reference frame because it will have to take into account just the interactions between material objects (which will not depend on location, orientation, and time instant). However, proving the existence of at least one Galilean reference frame is strictly impossible, hence it is taken as a principle: the ***principle of existence of a Galilean reference frame***.[3]

[3] In Newton's *Principia Mathematica*, Galilean reference frames do not appear explicitly. Instead, Newton shows the difference between *true motion* and *relative motion*, which correspond to motion with respect to a Galilean and to a non-Galilean reference frame, respectively.

## 1.3 Dynamics of a Free Particle: Newton's First Law (Principle of Inertia)

The simplest dynamical problem is that of a **free particle** (a particle free from interactions) observed from a Galilean reference frame. A free particle is an idealization: all real particles interact with other material objects located at a finite distance from them. Solving the dynamics of a free particle amounts to a thought experiment, but it is an interesting and useful one.

The properties of space and time in a Galilean reference frame partially restrict the possible motions of a free particle. If it is initially at rest, it will remain at rest (Fig. 1.2a): not doing so would imply choosing a direction for the initial motion, and that would contradict the isotropy of space. If its initial velocity is nonzero, its motion will have to be rectilinear (if it does not stop) though not necessarily uniform (Fig. 1.2b), as a curvature in its trajectory would again imply choosing a direction (the intrinsic normal direction). Summarizing: the properties of space and time in a Galilean reference frame (RGal) restrict the possible motion of the free particle according to $\mathrm{a}^{\mathrm{n}}_{\mathrm{RGal}}(\mathbf{P}) = 0$.

However, those properties do not imply that $\mathrm{a}^{\mathrm{s}}_{\mathrm{RGal}}(\mathbf{P}) = 0$. The particle could either increase or decrease its speed or even stop without contradicting any of them: this would be equivalent to imagine that space has a constant intrinsic friction equal in all locations and directions.

Stating that $\mathrm{a}^{\mathrm{s}}_{\mathrm{RGal}}(\mathbf{P}) = 0$, then, is formulating a principle (or a law): ***Newton's first law*** (***principle of inertia***). That principle is usually expressed mathematically as

$$\overline{\mathbf{a}}_{\mathrm{RGal}}(\mathbf{P}_{\mathrm{free}}) = \overline{0}. \quad (1.1)$$

This law is based on everyday experience, and had already been formulated (among others, by Galileo): if there are no interactions, rectilinear motion with constant speed lasts forever (Fig. 1.2b).

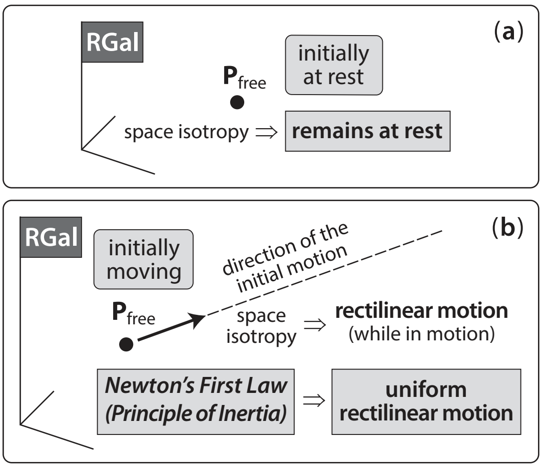


**Fig. 1.2**

The analytical expression of Newton's first law (Eq. (1.1)) shows that there is an infinite number of Galilean reference frames (provided that we accept the principle of existence of one Galilean reference frame): all those having a uniform translation relative to the postulated one. The proof is straightforward through a composition of accelerations: between any of those reference frames and the initial one, the transportation and the Coriolis accelerations are zero.

## 1.4 Dynamics of Interacting Particles

### Galileo's Principle of Relativity

We have just proved that all Galilean reference frames share the law governing the dynamics of a free particle.[4] When seeking for the law governing the dynamics of interacting particles, a logical question is: Will the Galilean reference frames also share that law? Answering that question is equivalent to establishing a principle of relativity (a principle defining the set of reference frames for which a scientific law is valid).

In Newtonian mechanics, that principle is the special principle of relativity, also known as ***Galileo's principle of relativity***, as it was first enunciated by Galileo in 1632. A modern formulation of that principle is: The laws governing all mechanical phenomena are the same in all reference frames where the ***Newton's first law*** is fulfilled. A consequence of that principle is that Galilean reference frames are not distinguishable through mechanical observations.[5]

### Mach's Empirical Propositions

Though the vector formulation of dynamics proposed by Newton is the most widespread one and its application is certainly more straightforward (as interaction forces are a more practical representation of interactions than interaction accelerations because of their symmetry, as will be seen later on), it has many drawbacks from an axiomatic point of view. It relies on the concepts *mass* and *force*, but the definition of the former is unclear and that of the latter is actually implicit.

The two axioms introduced so far (Newton's first law and the principle of relativity) do not mention mass or force, while Newton's second and third laws do explicitly

[4] Actually, Newton's first law provides an alternative definition for Galilean reference frames (initially defined as those where space is homogeneous and isotropic, and time is uniform): a reference frame R is Galilean when Newton's first law is fulfilled in R.

[5] This is actually how Galileo presents the principle: He proposes to "shut yourself up with some friend in the main cabin below decks on some large ship," proceed to observe several different mechanical phenomena, and then repeat the same observations but "have the ship proceed with any speed you like, so long as the motion is uniform and not fluctuating this way and that." He then affirms that "you will discover not the least change in all the effects named, nor could you tell from any of them whether the ship was moving or standing still," which is equivalent to saying that the laws governing the mechanical phenomena are exactly the same in all reference frames whose relative motion is a uniform translation (Galilei, G. [1953] *Dialogue Concerning the Two Chief World Systems*, trans. Stillman Drake).

include these concepts. Mach's formulation relies exclusively on interaction accelerations, and for this reason it will be presented before completing Newton's axioms.

Ernst Mach's axiomatization is based on three **empirical propositions** and two **definitions**. It is an alternative formulation whose application is not straightforward, but it has two important advantages:

- it is based on accelerations, which are observable (and measurable) variables, and for that reason they correspond to "real" concepts;
- it allows us to define in a rigorous way *mass* and *force*.

The **first empirical proposition** considers two isolated interacting particles **P** and **Q**. If observed from a Galilean reference frame, their accelerations will be exclusively the consequence of their mutual interaction. If $\overline{\mathbf{a}}_{\mathrm{RGal}}^{\mathbf{Q}\to\mathbf{P}}$ and $\overline{\mathbf{a}}_{\mathrm{RGal}}^{\mathbf{P}\to\mathbf{Q}}$ are the acceleration of **P** under the action of **Q** and that of **Q** under the action of **P**, respectively, the proposition states that those accelerations will be either attractive or repulsive, in the direction of $\overline{\mathbf{QP}}$ (Fig. 1.3). Mathematically, this can be expressed as

$$\frac{\overline{\mathbf{a}}_{\mathrm{RGal}}^{\mathbf{Q}\to\mathbf{P}}}{\left|\overline{\mathbf{a}}_{\mathrm{RGal}}^{\mathbf{Q}\to\mathbf{P}}\right|} = -\frac{\overline{\mathbf{a}}_{\mathrm{RGal}}^{\mathbf{P}\to\mathbf{Q}}}{\left|\overline{\mathbf{a}}_{\mathrm{RGal}}^{\mathbf{P}\to\mathbf{Q}}\right|}, \quad \overline{\mathbf{QP}} \times \overline{\mathbf{a}}_{\mathrm{RGal}}^{\mathbf{Q}\to\mathbf{P}} = \overline{0}. \tag{1.2}$$

This proposition is followed by the definition of **inertial mass-ratio** of two particles $(\mathrm{m}_{\mathbf{Q}}/\mathrm{m}_{\mathbf{P}})$:

$$\frac{\mathrm{m}_{\mathbf{Q}}}{\mathrm{m}_{\mathbf{P}}} \equiv \frac{\left|\overline{\mathbf{a}}_{\mathrm{RGal}}^{\mathbf{Q}\to\mathbf{P}}\right|}{\left|\overline{\mathbf{a}}_{\mathrm{RGal}}^{\mathbf{P}\to\mathbf{Q}}\right|}. \tag{1.3}$$

If we choose a particular value $\mathrm{m}_{\mathbf{P}}$ for a particle, the mass of the other particles may be determined straightaway.

Equation (1.3) provides an interpretation for the inertial mass. As the product $\mathrm{m}_{\mathbf{P}}\left|\overline{\mathbf{a}}_{\mathrm{RGal}}^{\mathbf{Q}\to\mathbf{P}}\right|$ is equal to $\mathrm{m}_{\mathbf{P}}\left|\overline{\mathbf{a}}_{\mathrm{RGal}}^{\mathbf{Q}\to\mathbf{P}}\right|$, it follows that the particle which acquires a higher acceleration has a lower mass, and vice versa. Hence, the inertial mass is a measure of the resistance of a particle to change its state of motion.

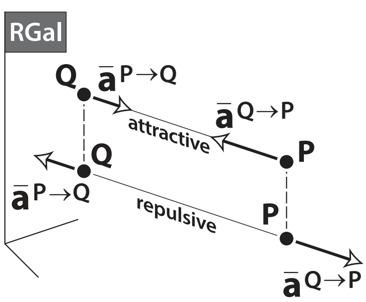


Fig. 1.3

The **second empirical proposition** is a principle of separation, and it guarantees that the determination of the ratio between the interaction accelerations of any pair of isolated particles (hence that of the mass-ratios) is unique through two statements:

A. the acceleration-ratio associated with a pair of particles is constant and independent from the interaction phenomenon

$$\frac{|\bar{\mathbf{a}}_{\mathrm{RGal}}^{\mathbf{Q}\to\mathbf{P}}|}{|\bar{\mathbf{a}}_{\mathrm{RGal}}^{\mathbf{P}\to\mathbf{Q}}|} = \text{constant} \Leftrightarrow \frac{m_{\mathbf{Q}}}{m_{\mathbf{P}}} = \text{constant}, \tag{1.4}$$

B. the acceleration-ratio is separable, that is, it can be expressed as the product of two independent acceleration-ratios.

This last idea is illustrated in Fig. 1.4 and can be formulated mathematically as:

$$\frac{|\bar{\mathbf{a}}_{\mathrm{RGal}}^{\mathbf{Q}\to\mathbf{P}}|}{|\bar{\mathbf{a}}_{\mathrm{RGal}}^{\mathbf{P}\to\mathbf{Q}}|} = \frac{|\bar{\mathbf{a}}_{\mathrm{RGal}}^{\mathbf{Q}\to\mathbf{S}}|}{|\bar{\mathbf{a}}_{\mathrm{RGal}}^{\mathbf{S}\to\mathbf{Q}}|} \cdot \frac{|\bar{\mathbf{a}}_{\mathrm{RGal}}^{\mathbf{S}\to\mathbf{P}}|}{|\bar{\mathbf{a}}_{\mathrm{RGal}}^{\mathbf{P}\to\mathbf{S}}|} \Leftrightarrow \frac{m_{\mathbf{Q}}}{m_{\mathbf{P}}} = \frac{m_{\mathbf{Q}}}{m_{\mathbf{S}}} \cdot \frac{m_{\mathbf{S}}}{m_{\mathbf{P}}}. \tag{1.5}$$

The **third empirical proposition** states that the accelerations that any number of particles (**Q**, **S**, **T**...) induce in a particle **P** are independent from each other. It is actually a principle of superposition, it is illustrated in Fig. 1.5 and can be formulated mathematically as:

$$\bar{\mathbf{a}}_{\mathrm{RGal}}^{(\mathbf{Q},\mathbf{S})\to\mathbf{P}} = \bar{\mathbf{a}}_{\mathrm{RGal}}^{\mathbf{Q}\to\mathbf{P}} + \bar{\mathbf{a}}_{\mathrm{RGal}}^{\mathbf{S}\to\mathbf{P}}. \tag{1.6}$$

The combination between the principle of separation and the principle of superposition for the interaction accelerations yields a principle of superposition for the mass of particles. If **QP** is the particle obtained when **P** and **Q** share the same location (two

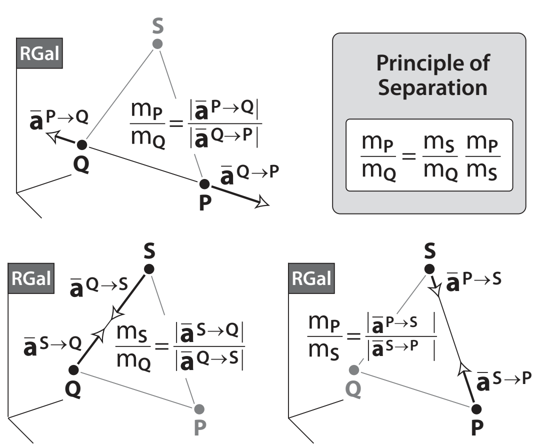


Fig. 1.4

dimensionless particles become one single particle in that case), its mass is the addition of those of **P** and **Q**: $m_{\mathbf{PQ}} = m_{\mathbf{P}} + m_{\mathbf{Q}}$.

Finally, Mach introduces the concept of interaction force through a definition:

$$\overline{F}_{\mathbf{Q}\to\mathbf{P}} \equiv m_{\mathbf{P}} \overline{\mathbf{a}}^{\mathbf{Q}\to\mathbf{P}}_{\mathrm{RGal}}. \tag{1.7}$$

The definition of interaction force is extremely useful: It describes the interaction between two particles through just one single magnitude $\left(\text{as } \left|\overline{F}_{\mathbf{Q}\to\mathbf{P}}\right| = \left|\overline{F}_{\mathbf{P}\to\mathbf{Q}}\right|\right)$. This is not possible when we use accelerations as main magnitude to describe that interaction: as the interacting particles acquire different accelerations $\left(\left|\overline{\mathbf{a}}^{\mathbf{Q}\to\mathbf{P}}_{\mathrm{RGal}}\right| \neq \left|\overline{\mathbf{a}}^{\mathbf{P}\to\mathbf{Q}}_{\mathrm{RGal}}\right|\right)$, we require two related magnitudes for a complete description (Fig. 1.6).

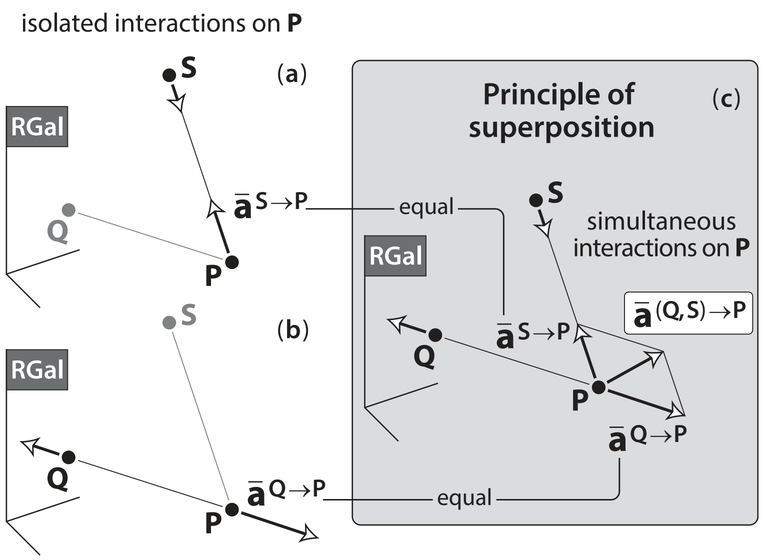


Fig. 1.5

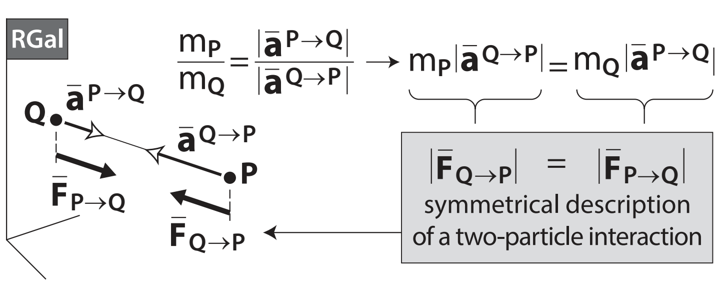


Fig. 1.6