THE ROAD TO PARADOX

Truth, provability, necessity, and other concepts are fundamental to many branches of philosophy, mathematics, computer science, and linguistics. Their study has led to some of the most celebrated achievements in logic, such as Gödel's incompleteness theorems, Tarski's theorem on the undefinability of truth, and numerous accounts of the paradoxes associated with these concepts. This book provides a clear and direct introduction to the theory of paradoxes and the Gödel incompleteness theorems. It offers new analyses of the ideas of self-reference, circularity, and the semantic paradoxes, and helps readers to see both how paradoxes arise and what their common features are. It will be valuable for students and researchers with a minimal background in logic and will equip them to understand and discuss a wide variety of topics in philosophical logic.

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A Guide to Syntax, Truth, and Modality

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CAMBRIDGE
UNIVERSITY PRESS

Shaftesbury Road, Cambridge CB2 8EA, United Kingdom

One Liberty Plaza, 20th Floor, New York, NY 10006, USA

477 Williamstown Road, Port Melbourne, VIC 3207, Australia

314–321, 3rd Floor, Plot 3, Splendor Forum, Jasola District Centre,
New Delhi – 110025, India

103 Penang Road, #05–06/07, Visioncrest Commercial, Singapore 238467

Cambridge University Press is part of Cambridge University Press & Assessment,
a department of the University of Cambridge.

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www.cambridge.org
Information on this title: www.cambridge.org/9781108841016

DOI: 10.1017/9781108888400

First published 2024

A catalogue record for this publication is available from the British Library

A Cataloging-in-Publication data record for this book is available from the Library of Congress

ISBN 978-1-108-84101-6 Hardback

In the beginning was the Word.
John 1:1

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Preface

This book has its origin in attempts to teach to philosophers the theory of the semantic paradoxes, formal theories of truth, and at least some ideas behind the Gödel incompleteness theorems. These are central topics in philosophical logic with many ramifications in other areas of philosophy and beyond. However, many texts on the paradoxes require an acquaintance with the theory of computation, the coding of syntax, and the representability of certain functions and relations in arithmetical theories. Teaching these techniques in class or covering them in an elementary text leaves little space for the actual topics, that is, the analysis of the paradoxes, formal theories of truth and other modalities, and the formalization of various metamathematical notions such as provability in a formal theory.

It is not necessary to learn about the theory of computation in order to understand even fairly subtle points about the semantic paradoxes, nor should it be. The paradoxes do not conceptually presuppose a coding of syntax in the natural numbers. In fact, the coding of syntax in the natural numbers can create artefacts that add an additional layer of complication, as we argue in the final chapter 12 of this book. Occasionally, logicians acknowledge this and precede their paper with the promise that their reasoning is not essentially based on arithmetic and could, or perhaps even should, be carried out directly in a theory of syntax without the detour through the coding of syntax. The authors of this book are among them. We honour our promise with this monograph. We show how arithmetic can be replaced in sophisticated theories of truth and parts of metamathematics; we even prove the Gödel incompleteness theorems in a syntax theory. Of course, for some applications the traditional arithmetized versions are required, but for others it is desirable to bypass the vagaries of coding and proceed in a more direct way. In this book we show that using a syntax theory does not impose limitations on the study of the paradoxes, formal theories of truth, and even some incompleteness phenomena.

We also fill a gap among the existing introductory texts in philosophical logic. The semantic paradoxes, self-reference, theories of truth and of modal predicates such as necessity, and issues related to the Gödel incompleteness theorems are at the centre of many discussions in philosophical logic. Yet these topics are shunned by many introductory texts in philosophical logic; and one might easily get the impression that philosophical logic is mainly concerned with nonclassical and intensional logics, including modal logic and its offspring such as deontic, dynamic, and epistemic logic. We provide an introduction to the theory of paradoxes that allows one to reach central results such as the diagonal lemma, Tarski's theory on the undefinability of truth, and the Yablo–Visser paradoxes relatively quickly. To provide precise proofs of both Gödel incompleteness theorems requires more effort, but we do this without going through coding and its mathematical details.

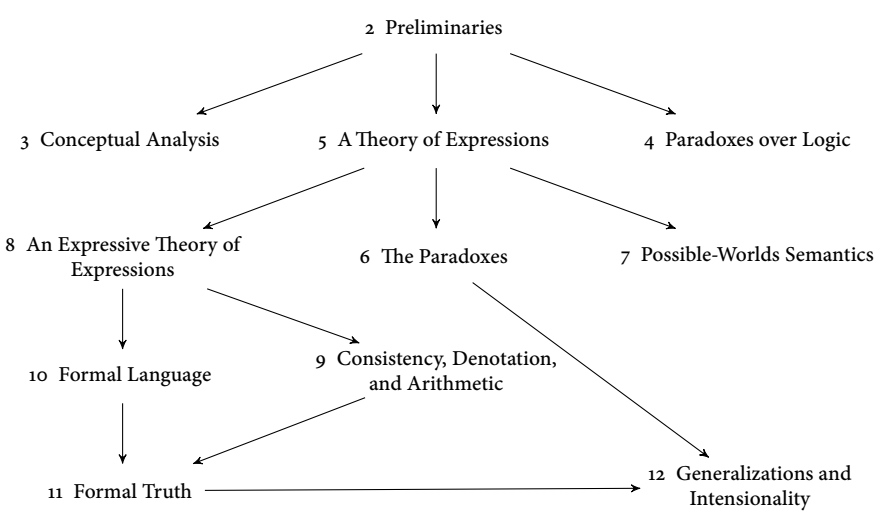
Intended Readership

We have written this book with a wide variety of readers in mind. It is aimed at an audience ranging from experts to less experienced readers who have covered only the basics of first-order predicate logic. The reader should be familiar at least with the material covered in an elementary logic textbook such as (Halbach 2010). For readers with a minimal background we have provided auxiliary material in chapter 2 which explains the notation and some topics such as function symbols that will be familiar to all readers with a more extensive background in logic.

In particular, the reader is not assumed to have seen proofs of the Gödel incompleteness theorems or to be at all familiar with the techniques involved in the proofs such as arithmetization and the theory of computability. Occasionally we establish connections with known logical and philosophical theories and results that are usually not covered in a basic logic course. For instance, in chapter 7 we look at a possible-worlds semantics that is modelled on possible-worlds semantics for modal logic; and in section 9.3 we show how theories of arithmetic can be recovered in a theory of expressions. These parts do not presuppose an acquaintance with modal logic or systems of arithmetic; but to appreciate their significance, a background in modal logic and formal systems of arithmetic is helpful. These parts are not built on in later sections of the book.

How to read this book

The book need not be read in a linear way. The following diagram displays the technical dependencies of the chapters. We hope that it is helpful especially when teaching from the book. For instance, the technical background required for the central chapter 6 on the paradoxes is covered in chapters 2 and 5. In particular, for a course on the formal theory of paradoxes and basic applications of diagonalization, students can be sent straight from chapter 2 to chapters 5 and 6. There is also a short path to the chapter on the Gödel incompleteness theorem, that is, chapter 10. An arrow from one chapter to another indicates that the chapter at the tip presupposes material from the chapter at the origin of the arrow.



Acknowledgements

We would like to thank our friends and colleagues for conversations and discussions related to the topics in this book, especially Balthasar Grabmayr, Leon Horsten, Ming Hsiung, Boaz Laan, Beau Mount, Carlo Nicolai, Lavinia Picollo, Wim Vanrie, Albert Visser, Philip Welch, Tim Williamson, Luke Wojtalik, Johanna Wolff, Lingyuan Ye, and an anonymous referee. The second author is forever grateful to Bahareh whose encouragement and support through stressful

times helped bring this project to fruition. Both authors are indebted to Christopher von Bülow for careful proofreading and eliminating many mistakes.

The New College Ludwig Fund for Humanities Research made it possible for Graham Leigh to visit Oxford in Trinity Term 2016 and to collaborate with Volker Halbach. Both authors would like to thank the New College Ludwig Fund, and Eugene Ludwig personally, for making this trip possible. We gratefully acknowledge support by the Arts & Humanities Research Council AH/H039791/1, the Leverhulme Trust, the Knut and Alice Wallenberg Foundation 2015.0179 and the Swedish Research Council 2017-05111.