

A Comparison Process for Mouse Pairs

This book proves some important new theorems in the theory of canonical inner models for large cardinal hypotheses, a topic of central importance in modern set theory. In particular, the author “completes” the theory of *Fine Structure and Iteration Trees* (FSIT) by proving a comparison theorem for mouse pairs parallel to the FSIT comparison theorem for pure extender mice, and then using the underlying comparison process to develop a fine structure theory for strategy mice.

Great effort has been taken to make the book accessible to non-experts so that it may also serve as an introduction to the higher reaches of inner model theory. It contains a good deal of background material, some of it unpublished folklore, and includes many references to the literature to guide further reading. An introductory essay serves to place the new results in their broader context.

This is a landmark work in inner model theory that should be in every set theorist’s library.

JOHN R. STEEL is Professor of Mathematics at the University of California, Berkeley. He is a recipient of the Carol Karp Prize of the Association for Symbolic Logic, the Hausdorff Medal of the European Set Theory Society, and the Humboldt Prize.

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***A Comparison Process
for Mouse Pairs***

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*For Colleen, David, Bobby,
and my parents*

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PREFACE

This book began life as a long research article titled *Normalizing iteration trees and comparing iteration strategies*. I found the main ideas behind the comparison process that motivates it in Spring 2015, and circulated a handwritten manuscript shortly afterward. I circulated a preliminary form of the present book in April 2016, and have revised and expanded it many times since then, as various significant gaps and errors showed up. The last major revisions took place in 2020-2021.

Beyond making the book correct, one of my goals has been to make it accessible. I have been encouraged here by the fact that the new definitions and results are actually quite elementary. They rest on the theory of *Fine structure and iteration trees* (FSIT), and can be seen as completing that theory in a certain way. The comparison theorem for pure extender mice that is at the heart of FSIT is deficient, in that how two mice compare depends on which iteration strategies are chosen to compare them. Here we remedy that defect, by developing a method for comparing the strategies. The result is a comparison theorem for *mouse pairs* parallel to the FSIT comparison theorem for pure extender mice. We then use the comparison process underlying that theorem to develop a fine structure theory for *strategy mice* parallel to the fine structure theory for pure extender mice of FSIT.

There are points at which descriptive set theory under determinacy hypotheses becomes relevant. At these points, it would help to have read the later sections of [70]. However, I have included enough material that the reader familiar with FSIT but shaky on determinacy should be able to follow the exposition. Our work here is motivated by the problem of analyzing ordinal definability in models of Axiom of Determinacy, but the prerequisite for following most of it is just inner model theory at the level of FSIT.

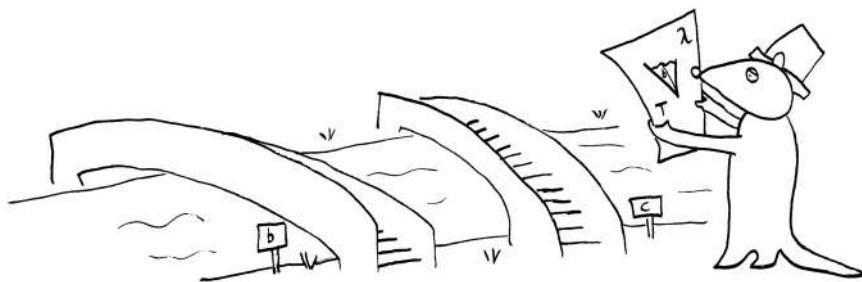
Inner model theory is sufficiently mature that many of its basic definitions and much of its notation have been streamlined and standardized over time. I have tried to contribute to that process, by adopting common definitions and notations where possible, and streamlining where I saw a chance to do so. Chapters 2 and 3 review and summarize the standard material on which the book rests, while Chapter 4 makes some revisions to it that are needed later.

The papers [77], [80], [78], [81], [64], [73], and [17], written in 2016-2018, have extended the work described here in various directions. The most important of

the remaining open problems is whether, assuming determinacy, there actually are mouse pairs at every appropriate level of logical complexity. This is one instance of the fundamental iterability problem of inner model theory, still open after all these years. I suspect that it is the most accessible instance of that problem.

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My larger intellectual debts are too numerous to list carefully. The generation before mine opened up the beautiful world of large cardinals, determinacy, and inner models, and many people have contributed to its development since then. Ronald Jensen, Tony Martin, and Bill Mitchell have had a special influence on my own work. In the immediate environment of this book, the work of Grigor Sargsyan and Hugh Woodin on the analysis of HOD in models of the Axiom of Determinacy plays an important role. My thanks to all these people.



A mouse with a strategy
Gregor Schindler, 2019