A Comparison Process for Mouse Pairs

This book proves some important new theorems in the theory of canonical inner models for large cardinal hypotheses, a topic of central importance in modern set theory. In particular, the author "completes" the theory of *Fine Structure and Iteration Trees* (FSIT) by proving a comparison theorem for mouse pairs parallel to the FSIT comparison theorem for pure extender mice, and then using the underlying comparison process to develop a fine structure theory for strategy mice.

Great effort has been taken to make the book accessible to non-experts so that it may also serve as an introduction to the higher reaches of inner model theory. It contains a good deal of background material, some of it unpublished folklore, and includes many references to the literature to guide further reading. An introductory essay serves to place the new results in their broader context.

This is a landmark work in inner model theory that should be in every set theorist's library.

JOHN R. STEEL is Professor of Mathematics at the University of California, Berkeley. He is a recipient of the Carol Karp Prize of the Association for Symbolic Logic, the Hausdorff Medal of the European Set Theory Society, and the Humboldt Prize.

LECTURE NOTES IN LOGIC

A Publication of The Association for Symbolic Logic

This series serves researchers, teachers, and students in the field of symbolic logic, broadly interpreted. The aim of the series is to bring publications to the logic community with the least possible delay and to provide rapid dissemination of the latest research. Scientific quality is the overriding criterion by which submissions are evaluated.

More information, including a list of the books in the series, can be found at http://aslonline.org/books/lecture-notes-in-logic/

LECTURE NOTES IN LOGIC 51

A Comparison Process for Mouse Pairs

JOHN R. STEEL
University of California, Berkeley

ASSOCIATION FOR SYMBOLIC LOGIC

CAMBRIDGE UNIVERSITY PRESS

Shaftesbury Road, Cambridge CB2 8EA, United Kingdom

One Liberty Plaza, 20th Floor, New York, NY 10006, USA

477 Williamstown Road, Port Melbourne, VIC 3207, Australia

314–321, 3rd Floor, Plot 3, Splendor Forum, Jasola District Centre,
New Delhi – 110025, India

103 Penang Road, #05–06/07, Visioncrest Commercial, Singapore 238467

Cambridge University Press is part of Cambridge University Press & Assessment,
a department of the University of Cambridge.

We share the University's mission to contribute to society through the pursuit of
education, learning and research at the highest international levels of excellence.

www.cambridge.org
Information on this title: www.cambridge.org/9781108840682

DOI: 10.1017/9781108886840

Association for Symbolic Logic
Richard A. Shore, Publisher
Department of Mathematics, Cornell University, Ithaca, NY 14853
http://aslonline.org

First published 2023

A catalogue record for this publication is available from the British Library.

ISBN 978-1-108-84068-2 Hardback

For Colleen, David, Bobby,
and my parents

CONTENTS

PREFACE

This book began life as a long research article titled *Normalizing iteration trees and comparing iteration strategies*. I found the main ideas behind the comparison process that motivates it in Spring 2015, and circulated a handwritten manuscript shortly afterward. I circulated a preliminary form of the present book in April 2016, and have revised and expanded it many times since then, as various significant gaps and errors showed up. The last major revisions took place in 2020-2021.

Beyond making the book correct, one of my goals has been to make it accessible. I have been encouraged here by the fact that the new definitions and results are actually quite elementary. They rest on the theory of *Fine structure and iteration trees* (FSIT), and can be seen as completing that theory in a certain way. The comparison theorem for pure extender mice that is at the heart of FSIT is deficient, in that how two mice compare depends on which iteration strategies are chosen to compare them. Here we remedy that defect, by developing a method for comparing the strategies. The result is a comparison theorem for *mouse pairs* parallel to the FSIT comparison theorem for pure extender mice. We then use the comparison process underlying that theorem to develop a fine structure theory for *strategy mice* parallel to the fine structure theory for pure extender mice of FSIT.

There are points at which descriptive set theory under determinacy hypotheses becomes relevant. At these points, it would help to have read the later sections of [70]. However, I have included enough material that the reader familiar with FSIT but shaky on determinacy should be able to follow the exposition. Our work here is motivated by the problem of analyzing ordinal definability in models of Axiom of Determinacy, but the prerequisite for following most of it is just inner model theory at the level of FSIT.

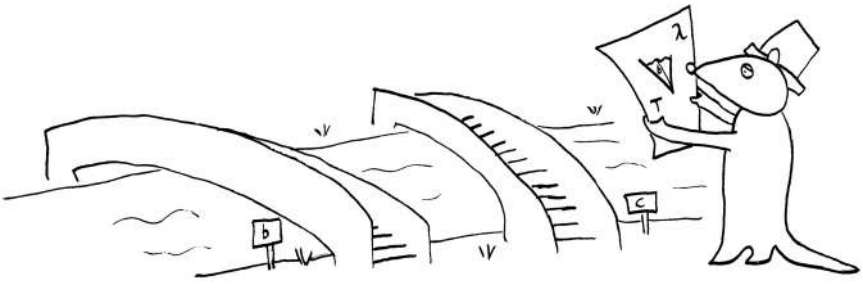
Inner model theory is sufficiently mature that many of its basic definitions and much of its notation have been streamlined and standardized over time. I have tried to contribute to that process, by adopting common definitions and notations where possible, and streamlining where I saw a chance to do so. Chapters 2 and 3 review and summarize the standard material on which the book rests, while Chapter 4 makes some revisions to it that are needed later.

The papers [77], [80], [78], [81], [64], [73], and [17], written in 2016-2018, have extended the work described here in various directions. The most important of

the remaining open problems is whether, assuming determinacy, there actually are mouse pairs at every appropriate level of logical complexity. This is one instance of the fundamental iterability problem of inner model theory, still open after all these years. I suspect that it is the most accessible instance of that problem.

Acknowledgements. Special thanks to Xianghui Shi, who converted my handwritten manuscript to a LaTeX document in summer 2015, and thereby helped me over the "getting started" barrier. I worked on the first draft in Fall 2015 as a Simons Foundation fellow at the Isaac Newton Institute for Mathematical Sciences in the programme 'Mathematical, Foundational and Computational Aspects of the Higher Infinite'. My thanks to the institute and its staff for their support. Thanks also to the participants in the annual summer workshops in inner model theory, who helped me clarify the material here (and inspired the drawing below). Special thanks to Benjamin Siskind for his help in identifying and correcting some nontrivial errors. Finally, thanks to Nam Trang for his input at various stages, including help with tikzpicture, to Thomas Piecha for his help with formatting, to the referees for their input, and to Paul Larson for his work as editor.

My larger intellectual debts are too numerous to list carefully. The generation before mine opened up the beautiful world of large cardinals, determinacy, and inner models, and many people have contributed to its development since then. Ronald Jensen, Tony Martin, and Bill Mitchell have had a special influence on my own work. In the immediate environment of this book, the work of Grigor Sargsyan and Hugh Woodin on the analysis of HOD in models of the Axiom of Determinacy plays an important role. My thanks to all these people.



A mouse with a strategy
Gregor Schindler, 2019