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John R. Steel
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Chapter 1

INTRODUCTION

In this book we shall develop a general comparison process for iteration strategies, and show how the process can be used to analyze ordinal definability in models of the Axiom of Determinacy. In this introduction, we look at the context and motivation for the technical results to come.

We begin with a broad overview of *inner model theory*, the subject to which this book belongs. Eventually we reach an outline of the ideas and results that are new here. The journey is organized so that the technical background needed to follow along increases as we proceed.

1.1. Large cardinals and the consistency strength hierarchy

Strong axioms of infinity, or as they are more often called, large cardinal hypotheses, play a central role in set theory. There are at least two reasons.

First, large cardinal hypotheses can be used to decide in a natural way many questions which cannot be decided on the basis of ZFC (the commonly accepted system of axioms for set theory, and hence all of mathematics). Many such questions come from *descriptive set theory*, the theory of simply definable sets of real numbers. For example, the hypothesis that there are infinitely many Woodin cardinals yields a systematic and detailed theory of the *projective* sets of reals, those that are definable in the language of second order arithmetic from real parameters. ZFC by itself yields such a theory at only the simplest levels of second order definability.

Second, large cardinal hypotheses provide a way of organizing and surveying all possible natural extensions of ZFC. This is due to the following remarkable phenomenon: for any natural extension T of ZFC which set theorists have studied, there seems to be an extension S of ZFC axiomatized by large cardinal hypotheses such that the consistency of T is provably (in ZFC) equivalent to that of S. The consistency strengths of the large cardinal hypotheses are linearly ordered, and usually easy to compare. Thus all natural extensions of ZFC seem to fall into a hierarchy linearly ordered by consistency strength, and calibrated by the large cardinal hypotheses.[1]

[1] Let $\mathrm{con}(T)$ be some natural formalization of the assertion that T is consistent. The consistency strength order is given by: $S \leq_{\mathrm{con}} T$ iff ZFC proves $\mathrm{con}(T) \rightarrow \mathrm{con}(S)$.

These two aspects of large cardinal hypotheses are connected, in that the consistency strength order on natural theories corresponds to the inclusion order on the set of their "sufficiently absolute" consequences. For example, if S and T are natural theories extending ZFC, and S has consistency strength less than or equal to that of T, then the arithmetic consequences of S are included in those of T. If in addition, S and T have consistency strength at least that of "there are infinitely many Woodin cardinals", then the consequences of S in the language of second order arithmetic are included in those of T. This pattern persists at still higher consistency strengths, with still more logically complicated consequences about reals and sets of reals being brought into a uniform order. This beautiful and suggestive phenomenon has a practical dimension as well: one way to develop the absolute consequences of a strong theory T is to compute a consistency strength lower bound S for T in terms of large cardinal hypotheses, and then work in the theory S. For one of many examples, the Proper Forcing Axiom (PFA) yields a canonical inner model with infinitely many Woodin cardinals that is correct for statements in the language of second order arithmetic, and therefore PFA implies all consequences of the existence of infinitely many Woodin cardinals that can be stated in the language of second order arithmetic.

One can think of the consistency strength of a theory as the degree to which it is committed to the existence of the higher infinite. Large cardinal hypotheses make their commitments explicitly: they simply say outright that the infinities in question exist. It is therefore usually easy to compare their consistency strengths. Other natural theories often have their commitments to the existence of the infinite well hidden. Nevertheless, set theorists have developed methods whereby these commitments can be brought to the surface, and compared. These methods have revealed the remarkable phenomenon described in the last paragraph, that natural theories appear to be wellordered by the degrees to which they are committed to the infinite, and that this degree of commitment corresponds exactly to the power of the theory to decide questions about concrete objects, like natural numbers, real numbers, or sets of real numbers.

We should emphasize that the paragraphs above describe a general pattern of existing theorems. There are many examples of natural theories whose consistency strengths have not yet been computed, and perhaps they, or some natural theory yet to be found, will provide counterexamples to the pattern described above. The pervasiveness of the pattern where we know how to compare consistency strengths is evidence that this will not happen.[2] The two methods whereby set theorists compare consistency strengths, forcing and inner model theory, seem to lead inevitably to the pattern. In particular, the wellorder of natural consistency strengths seems to correspond to the inclusion order on canonical minimal inner

[2]The pattern extends to weak subtheories of ZFC as well. This book is concerned only with theories having very strong commitments to infinity, and so we shall ignore subtheories of ZFC, but the linearity of the consistency strengths below that of ZFC is evidence of linearity higher up.

models for large cardinal hypotheses. Forcing and inner model theory seem sufficiently general to compare all natural consistency strengths, but at the moment, this is just informed speculation. So one reasonable approach to understanding the general pattern of consistency strengths is to develop our comparison methods further. In particular, inner model theory is in great need of further development, as there are quite important consistency strengths that it does not yet reach.

1.2. Inner model theory

The inner model program attempts to associate to each large cardinal hypothesis H a canonical minimal universe of sets M_H (an *inner model*) in which H is true. The stronger H is, the larger M_H will be; that is, $G \leq_{\text{con}} H$ if and only if $M_G \subseteq M_H$. Some of our deepest understanding of large cardinal hypotheses comes from the inner model program.

The inner models we have so far constructed have an internal structure which admits a systematic, detailed analysis, a *fine structure theory* of the sort pioneered by Ronald Jensen around 1970 ([18]). Thus being able to construct M_H gives us a very good idea as to what a universe satisfying H might look like. Inner model theory thereby provides evidence of the consistency of the large cardinal hypotheses to which it applies. (The author believes that this will some day include all the large cardinal hypotheses currently studied.) Since forcing seems to reduce any consistency question to the consistency question for some large cardinal hypothesis, it is important to have evidence that the large cardinal hypotheses themselves are consistent! No evidence is more convincing than an inner model theory for the hypothesis in question.

The smallest of the canonical inner models is the universe L of constructible sets, isolated by Kurt Gödel ([14]) in his 1937 proof that CH is consistent with ZFC. It was not until the mid 1960s that J. Silver and K. Kunen ([62], [27]) developed the theory of a canonical inner model going properly beyond L, by constructing M_H for $H =$ “there is a measurable cardinal”.[3] Since then, progressively larger M_H for progressively stronger H have been constructed and studied in detail. (See for example [5], [31], and [32].) At the moment, we have a good theory of canonical inner models satisfying “there is a Woodin cardinal”, and even slightly stronger hypotheses. (See [30], [35], and [66], for example.) One of the most important open problems in set theory is to extend this theory significantly further, with perhaps the most well-known target being models satisfying “there is a supercompact cardinal”.

[3] ZFC is of course too weak, consistency-wise, to prove that there is such a model. Silver and Kunen worked in the theory $\mathsf{ZFC}+$ “there is a measurable cardinal”. In the mid 1970s, Dodd and Jensen developed general methods for constructing the canonical inner model with a measurable under a wide assortment of hypotheses. See [5], [6], and [7].

Inner model theory is a crucial tool in calibrating consistency strengths: in order to prove that $H \leq_{\text{con}} T$, where H is a large cardinal hypothesis, one generally constructs a canonical inner model of H inside an arbitrary model of T. Because we do not have a full inner model theory very far past Woodin cardinals, we lack the means to prove many well-known conjectures of the form $H \leq_{\text{con}} T$, where H is significantly stronger than "there is a Woodin cardinal". Broadly speaking, there are great defects in our understanding of the consistency strength hierarchy beyond Woodin cardinals.

Inner model theory is also a crucial tool in developing the consequences for real numbers of large cardinal hypotheses. Indeed, the basics of inner model theory for Woodin cardinals were discovered in 1985-86 by D. A. Martin and the author, at roughly the same time they discovered their proof of Projective Determinacy, or PD. (Martin, Moschovakis, and others had shown in the 1960s and 70s that PD decides in a natural way all the classical questions about projective sets left undecided by ZFC alone.) This simultaneous discovery was not an accident, as the fundamental new tool in both contexts was the same: *iteration trees*, and the *iteration strategies* which produce them. Since then, progress in inner model theory has given us a deeper understanding of pure descriptive set theory, and the means to solve some old problems in that field.

The fundamental open problem of inner model theory is to extend the theory to models satisfying stronger large cardinal hypotheses. "There is a supercompact cardinal" is an old and still quite challenging target. One very well known test question here is whether $(\mathsf{ZFC}+$"there is a supercompact cardinal"$) \leq_{\text{con}} \mathsf{ZFC}+\mathsf{PFA}$. The answer is almost certainly yes, and the proof almost certainly involves an inner model theory that is firing on all cylinders.[4] That kind of inner model theory we have now only at the level of many Woodin cardinals, but significant parts of the theory do exist already at much higher levels.[5]

1.3. Mice and iteration strategies

The canonical inner models we seek are often called *mice*. There are two principal varieties, the pure extender mice and the strategy mice.[6]

[4]A parallel, and still older, question is whether $(\mathsf{ZFC}+$"there is a supercompact cardinal"$) \leq_{\text{con}}$ $\mathsf{ZFC}+$"there is a strongly compact cardinal".

[5]J. Baumgartner showed in the early 1980s that $\mathsf{ZFC}+\mathsf{PFA} \leq_{\text{con}} \mathsf{ZFC}+$"there is a supercompact cardinal". Supercompacts are far beyond Woodin cardinals, in the sense that there are many interesting consistency strengths strictly between the two, and in the sense that constructing canonical inner models for supercompacts presents significant new difficulties. Many set theoretic principles have been shown consistent relative to the existence of (sometimes many) supercompact cardinals, so inner-model-theoretic evidence of their consistency would be valuable.

[6]Strategy mice are sometimes called *hod mice*, because of their role in analyzing the hereditarily ordinal definable sets in models of the Axiom of Determinacy.

A pure extender premouse is a model of the form $L_\alpha[\vec{E}]$ where $\vec{E}$ is a coherent sequence of extenders. Here an extender is a system of ultrafilters coding an elementary embedding, and coherence means roughly that the extenders appear in order of strength, without leaving gaps. These notions were introduced by Mitchell in the 1970s[7], and they have been a foundation for work in inner model theory since then.

In this book, we shall assume that our premice have no long extenders on their coherent sequences.[8] Such premice can model superstrong, and even subcompact, cardinals. They cannot model κ^+-supercompactness. Long extenders lead to an additional set of difficulties.

An *iteration strategy* is a winning strategy for player II in the iteration game. For any premouse M, the iteration game on M is a two player game of length $\omega_1 + 1$.[9] In this game, the players construct a tree of models such that each successive node on the tree is obtained by an ultrapower of a model that already exists in the tree. I is the player that describes how to construct this ultrapower. He chooses an extender E from the sequence of the last model N constructed so far, then chooses another model P in the tree and takes the ultrapower of P by E. If the ultrapower is ill-founded then player I wins; otherwise the resulting ultrapower is the next node on the tree. Player II moves at limit stages λ by choosing a branch of the tree that has been visited cofinally often below λ, and is such that the direct limit of the embeddings along the branch is well-founded. If he fails to do so, he loses. If II manages to stay in the category of wellfounded models through all $\omega_1 + 1$ moves, then he wins. A winning strategy for II in this game is called an *iteration strategy* for M, and M is said to be *iterable* just in case there is an iteration strategy for it. Iterable pure extender premice are called *pure extender mice*.

Pure extender mice are canonical objects; for example, any real number belonging to such a mouse is ordinal definable. Let us say that a premouse M is *pointwise definable* if every element of M is definable over M. For any axiomatizable theory T, the minimal mouse satisfying T is pointwise definable. The canonicity of pure extender mice is due to their iterability, which, via the fundamental *Comparison Lemma*, implies that the pointwise definable pure extender mice are wellordered by inclusion. This is the *mouse order* on pointwise definable pure extender mice. The consistency strength of T is determined by the minimal mouse M having a generic extension satisfying T, and thus the consistency strength order on natural T is mirrored in the mouse order. However, in the case of the mouse order, we have *proved* that we have a wellorder; what we cannot yet do is tie natural T at high consistency strengths to it. As we climb the mouse order, the mice become

[7]See [31] and [32].

[8]An extender is short if all its component ultrafilters concentrate on the critical point. Otherwise, it is long.

[9]Iteration games of other lengths are also important, but this length is crucial, so we shall focus on it.

correct (reflect what is true in the full universe of sets) at higher and higher levels of logical complexity.

Iteration strategies for pointwise definable pure extender mice are also canonical objects; for example, a pointwise definable mouse has exactly one iteration strategy.[10] The existence of iteration strategies is at the heart of the fundamental problem of inner model theory, and for a pointwise definable M, to prove the existence of an iteration strategy is to define it. In practice, it seems necessary to give a definition in the simplest possible logical form. As we go higher in the mouse order, the logical complexity of iteration strategies must increase, in a way that keeps pace with the correctness of the mice they identify.

Our most powerful, all-purpose method for constructing iteration strategies is the *core model induction method*. Because iteration strategies must act on trees of length ω_1, they are not coded by sets of reals. Nevertheless, the fragment of the iteration strategy for a countable mouse that acts on countable iteration trees *is* coded by a set of reals. If this set happens to be absolutely definable (that is, Universally Baire) then the strategy can be extended to act on uncountable iteration trees in a unique way. There is no other way known to construct iteration strategies acting on uncountable trees. Thus, having an absolutely definable iteration strategy for countable trees is tantamount to having a full iteration strategy. The key idea in the core model induction is to use the concepts of descriptive set theory, under determinacy hypotheses, to identify a next relevant level of correctness and definability for sets of reals, a target level at which the next iteration strategy should be definable.

Absolute definability leads to determinacy. Thus at reasonably closed limit steps in a core model induction, one has a model M of $\mathsf{AD}+V=L(P(\mathbb{R}))$ that contains the restrictions to countable trees of the iteration strategies already constructed. Understanding the structure of HOD^M is important for going further.

1.4. HOD in models of determinacy

HOD is the class of all hereditarily ordinal definable sets. It is a model of ZFC[11], but beyond that, ZFC does not decide its basic theory, and the same is true of ZFC augmented by any of the known large cardinal hypotheses. The problem is that the definitions one has allowed are not sufficiently absolute. In contrast, the theory of HOD in determinacy models is well-determined, not subject to the vagaries of forcing.[12]

[10] This follows from Theorem 4.11 of [70], and the fact that any iteration strategy for a pointwise definable M has the Weak Dodd-Jensen property with respect to all enumerations of M.

[11] See [36].

[12] We mean here determinacy models of the form $M=L(\Gamma,\mathbb{R})$, where Γ is a proper initial segment of the universally Baire sets. If there are arbitrarily large Woodin cardinals, then for any sentence φ,

The study of HOD in models of AD has a long history. The reader should see [72] for a survey of this history. HOD was studied by purely descriptive set theoretic methods in the late 70s and 80s, and partial results on basic questions such as whether HOD $\models$ GCH were obtained then. It was known then that inner model theory, if only one could develop it in sufficient generality, would be relevant to characterizing the reals in HOD. It was known that HOD^M is close to M in various ways; for example, if $M \models \mathsf{AD}^+ + V = L(P(\mathbb{R}))$[13], then M can be realized as a symmetric forcing extension of HOD^M, so that the first order theory of M is part of the first order theory of its HOD.[14]

Just how relevant inner model theory is to the study of HOD in models of AD became clear in 1994, when the author showed that if there are ω Woodin cardinals with a measurable above them all, then $\mathrm{HOD}^{L(\mathbb{R})}$ up to $\theta^{L(\mathbb{R})}$ is a pure extender mouse.[15](See [65].) Shortly afterward, this result was improved by Hugh Woodin, who reduced its hypothesis to $\mathsf{AD}^{L(\mathbb{R})}$, and identified the full $\mathrm{HOD}^{L(\mathbb{R})}$ as a model of the form $L[M,\Sigma]$, where M is a pure extender premouse, and Σ is a partial iteration strategy for M. $\mathrm{HOD}^{L(\mathbb{R})}$ is thus a new type of mouse, sometimes called a *strategy mouse*, sometimes called a *hod mouse*. See [82] for an account of this work.

Since the mid-1990s, there has been a great deal of work devoted to extending these results to models of determinacy beyond $L(\mathbb{R})$. Woodin analyzed HOD in models of AD^+ below the minimal model of $\mathsf{AD}_\mathbb{R}$ fine structurally, and Sargsyan extended the analysis further, first to determinacy models below $\mathsf{AD}_\mathbb{R} +$ "θ is regular" (see [42] and [43]), and more recently, to models of still stronger forms of determinacy.[16] Part of the motivation for this work is that it seems to be essential in the core model induction: in general, the next iteration strategy seems to be a strategy for a hod mouse, not for a pure extender mouse. This idea comes from work of Woodin and Ketchersid around 2000. (See [25] and [52].)

1.5. Least branch hod pairs

The strategy mice used in the work just described have the form $M = L[\vec{E},\Sigma]$, where $\vec{E}$ is a coherent sequence of extenders, and Σ is an iteration strategy for M. The strategy information is fed into the model M slowly, in a way that is dictated

whether φ is true in all such HOD^M is absolute under set forcing. This follows easily from Woodin's theorem on the generic absoluteness of $(\Sigma^2_1)^{\mathrm{uB}}$ statements. See [69, Theorem 5.1].

[13] AD^+ is a technical strengthening of AD. It is not known whether $\mathsf{AD} \Longrightarrow \mathsf{AD}^+$, but in every model of AD constructed so far, AD^+ also holds. In particular, the models of AD that are relevant in the core model induction satisfy AD^+.

[14] This is a theorem of Woodin from the early 1980s. Cf. [72].

[15] In a determinacy context, θ denotes the least ordinal that is not the surjective image of the reals.

[16] See [44]. Part of this work was done in collaboration with the author; see [75], [79], and [76]. The determinacy principles dealt with here are all weaker than a Woodin limit of Woodin cardinals.

in part by the determinacy model whose HOD is being analyzed. One says that the hierarchy of M is *rigidly layered*, or *extender biased*. The object (M,Σ) is called a rigidly layered (extender biased) *hod pair*.

Perhaps the main motivation for the extender biased hierarchy is that it makes it possible to prove a comparison theorem. There is no inner model theory without such a theorem. Comparing strategy mice necessarily involves comparing iteration strategies, and comparing iteration strategies is significantly more difficult than comparing extender sequences. Rigid layering lets one avoid the difficulties inherent in the general strategy comparison problem, while proving comparison for a class of strategy mice adequate to analyze HOD in the minimal model of $\mathsf{AD}_{\mathbb{R}}$ + "θ is regular", and somewhat beyond. The key is that in this region, HOD does not have cardinals that are strong past a Woodin cardinal.

Unfortunately, rigid layering does not seem to help in comparing strategy mice that have cardinals that are strong past a Woodin. Moreover, it has serious costs. The definition of "hod premouse" becomes very complicated, and indeed it is not clear how to extend the definition of rigidly layered hod pairs much past that given in [44]. The definition of "rigidly layered hod premouse" is not uniform, in that the extent of extender bias depends on the determinacy model whose HOD is being analyzed. Fine structure, and in particular condensation, become more awkward. For example, it is not true in general that the pointwise definable hull of a level of M is a level of M. (The problem is that the hull will not generally be sufficiently extender biased.)

The more naive notion of hod premouse would abandon extender bias, and simply add the least missing piece of strategy information at essentially every stage. This was originally suggested by Woodin.[17] The focus of this book is a general comparison theorem for iteration strategies that makes it possible to use this approach, at least in the realm of short extenders. The resulting premice are called *least branch premice* (lpm's), and the pairs (M,Σ) are called *least branch hod pairs* (lbr hod pairs). Combining results of this book and [73], one has

THEOREM 1.5.1 ([73]). *Assume* AD^+ + *"there is an* (ω_1,ω_1) *iteration strategy for a pure extender premouse with a long extender on its sequence"; then*

(1) for any $\Gamma \subseteq P(\mathbb{R})$ *such that* $L(\Gamma,\mathbb{R}) \models \mathsf{AD}_{\mathbb{R}}$ + *"there is no* (ω_1,ω_1) *iteration strategy for a pure extender premouse with a long extender on its sequence",* $\mathrm{HOD}^{L(\Gamma,\mathbb{R})}$ *is a least branch premouse, and*

(2) there is a $\Gamma \subseteq P(\mathbb{R})$ *such that* $L(\Gamma,\mathbb{R}) \models \mathsf{AD}_{\mathbb{R}}$ + *"there is no* (ω_1,ω_1) *iteration strategy for a pure extender premouse with a long extender on its sequence", and* $\mathrm{HOD}^{L(\Gamma,\mathbb{R})} \models$ *"there is a subcompact cardinal".*

[17]There are some fine-structural problems with the precise method for inserting strategy information originally suggested by Woodin. The method for strategy insertion that is correct in detail is due to Schlutzenberg and Trang. Cf. [61].

Of course, one would like to remove the mouse existence hypothesis of Theorem 1.5.1, and prove its conclusion under AD^+ alone. Finding a way to do this is one manifestation of the long standing iterability problem we have discussed above. Although we do not yet know how to do this, the theorem does make it highly likely that in models of $\mathsf{AD}_\mathbb{R}$ that have not reached an iteration strategy for a pure extender premouse with a long extender, HOD is a least branch premouse. It also makes it very likely that there are such HOD's with subcompact cardinals. Subcompactness is one of the strongest large cardinal properties that can be represented with short extenders.[18]

Although we shall not prove Theorem 1.5.1 here, we shall prove an approximation to it that makes the same points. That approximation is Theorem 11.3.13 below.

Least branch premice have a fine structure much closer to that of pure extender models than that of rigidly layered hod premice. In this book we develop the basics, including the solidity and universality of standard parameters, and a form of condensation. In [81], the author and N. Trang have proved a sharper condensation theorem, whose pure extender version was used heavily in the Schimmerling-Zeman work ([49]) on $\square$ in pure extender mice. It seems likely that the rest of the Schimmerling-Zeman work extends as well.

Thus least branch hod pairs give us a good theory of HOD in the short extender realm, provided there are enough such pairs.[19] Below, we formulate a conjecture that we call *Hod Pair Capturing*, or HPC, that makes precise the statement that there are enough least branch hod pairs. HPC is the main open problem in the theory to which this book contributes.

1.6. Comparison and the mouse pair order

Let us first say more about the nature of least branch hod pairs (M,Σ). There are four requirements on Σ in the definition: strong hull condensation, quasi-normalizing well, internal lift consistency, and pushforward consistency. We shall describe these requirements informally, omitting some of the fine points, and give the full definitions later.

Recall that an iteration tree on a premouse M is *normal* iff the extenders $E_\alpha^\mathcal{W}$ used in $\mathcal{W}$ have lengths increasing with α, and each $E_\alpha^\mathcal{W}$ is applied to the longest possible initial segment of the earliest possible model in $\mathcal{W}$. For technical reasons we need to consider a slight weakening of the length-increasing requirement; we call the resulting trees *quasi-normal*. Our iteration strategies will act on finite *stacks* of quasi-normal trees, that is, sequences $s = \langle \mathcal{T}_0, \dots, \mathcal{T}_n \rangle$ such that for all

[18] Until now, there was no very strong evidence that the HOD of a determinacy model could satisfy that there are cardinals that are strong past a Woodin cardinal.

[19] At least in the case that the background determinacy model satisfies $\mathsf{AD}_\mathbb{R} + V = L(P(\mathbb{R}))$. Some form of extender bias may be appropriate in other cases.

$k \leq n-1$, $\mathcal{T}_{k+1}$ is a quasi-normal tree on some initial segment of the last model in $\mathcal{T}_k$. We write $M_\infty(s)$ for the last model of $\mathcal{T}_n$, if there is one.

DEFINITION 1.6.1. Let Σ be an iteration strategy for a premouse P.

(1) (Tail strategy) If s is a stack by Σ and $Q \trianglelefteq M_\infty(s)$, then $\Sigma_{s,Q}$ is the strategy for Q given by: $\Sigma_{s,Q}(t) = \Sigma(s^\frown\langle Q,t\rangle)$.[20]
(2) (Pullback strategy) If $\pi\colon N \to P$ is elementary, then Σ^π is the strategy for N given by: $\Sigma^\pi(s) = \Sigma(\pi s)$, where πs is the lift of s by π to a stack on P.

In (2), elementarity must be understood fine structurally; our convention is that every premouse P has a degree of soundness attached to it, and elementarity means elementarity at that quantifier level.

Perhaps the most important regularity property of iteration strategies is *strong hull condensation*. To define it we need the notion of a *tree embedding* $\Phi\colon \mathcal{T} \to \mathcal{U}$, where $\mathcal{T}$ and $\mathcal{U}$ are normal trees on the same M. The idea of course is that Φ should preserve a certain amount of the iteration tree structure, but some care is needed in spelling out exactly how much. Φ is determined by a map $u\colon \operatorname{lh}(\mathcal{T}) \to \operatorname{lh}(\mathcal{U})$ and maps $\pi_\alpha\colon \mathcal{M}^{\mathcal{T}}_\alpha \to \mathcal{M}^{\mathcal{U}}_{u(\alpha)}$ having various properties. See Section 6.4.

DEFINITION 1.6.2. Let Σ be an iteration strategy for a premouse M; then Σ has *strong hull condensation* iff whenever s is a stack of normal trees by Σ and $N \trianglelefteq \mathcal{M}_\infty(s)$, and $\mathcal{U}$ is a normal tree on N by $\Sigma_{s,N}$, and $\Phi\colon \mathcal{T} \to \mathcal{U}$ is a tree embedding, with associated maps $\pi_\alpha\colon \mathcal{M}^{\mathcal{T}}_\alpha \to \mathcal{M}^{\mathcal{U}}_{u(\alpha)}$, then

(a) $\mathcal{T}$ is by $\Sigma_{s,N}$, and
(b) for all $\alpha < \operatorname{lh}(\mathcal{T})$, $\Sigma_{s^\frown\langle N,\mathcal{T}\restriction\alpha+1\rangle} = (\Sigma_{s^\frown\langle N,\mathcal{U}\restriction u(\alpha)+1\rangle})^{\pi_\alpha}$.

Strong hull condensation is a stronger version of the hull condensation property isolated by Sargsyan in [42].

The second important property is quasi-normalizing well. Given a stack $\langle \mathcal{T},\mathcal{U}\rangle$ on M with last model N such that $\mathcal{T}$ and $\mathcal{U}$ are normal, shuffling the extenders of $\mathcal{U}$ into $\mathcal{T}$ in a minimal way produces a normal tree $\mathcal{W} = W(\mathcal{T},\mathcal{U})$. If $\mathcal{U}$ has a last model R, we get a nearly elementary map $\pi\colon N \to R$. We call $W(\mathcal{T},\mathcal{U})$ the *embedding normalization* of $\langle \mathcal{T},\mathcal{U}\rangle$. The idea is simple, but there are many technical details.[21] It proves useful to consider a slightly less minimal shuffling $V(\mathcal{T},\mathcal{U})$ that we call the *quasi-normalization* of $\langle \mathcal{T},\mathcal{U}\rangle$. Even if $\mathcal{T}$ and $\mathcal{U}$ are normal, $V(\mathcal{T},\mathcal{U})$ may not be length-increasing, but it is nearly so. The reader should see Chapter 6 for full definitions.

DEFINITION 1.6.3. Let Σ be an iteration strategy for a premouse M. We say that Σ *quasi-normalizes well* iff whenever s is an stack on M by Σ, and $\langle \mathcal{T},\mathcal{U}\rangle$ is a stack of length 2 by Σ_s such that $\mathcal{T}$ and $\mathcal{U}$ are normal trees having last models, then

[20] For premice Q and R, $Q \trianglelefteq R$ iff the hierarchy of Q is an initial segment of that of R.
[21] Much of the general theory of normalization was developed independently by Schlutzenberg. See [57]. See also [21] and [63].