Galois Theory and Its Algebraic Background

SECOND EDITION

Galois theory, the theory of polynomial equations and their solutions, is one of the most fascinating and beautiful subjects in pure mathematics. Using group theory and field theory, it provides a complete answer to the problem of the solubility of polynomial equations by radicals: that is, determining when and how a polynomial equation can be solved by repeatedly extracting roots using elementary algebraic operations.

This textbook contains a fully detailed account of Galois theory and the algebra that it needs, and is suitable for both those following a course of lectures and the independent reader (who is assumed to have no previous knowledge of Galois theory). This second edition has been significantly revised and reordered; the first part develops the basic algebra that is needed, and the second part gives a comprehensive account of Galois theory. There are applications to ruler and compass constructions, and to the solution of classical mathematical problems of ancient times. There are new exercises throughout, and carefully selected examples will help the reader develop a clear understanding of the mathematical theory.

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Preface

Galois theory is one of the most fascinating and enjoyable branches of algebra. The problems with which it is concerned have a long and distinguished history: the problems of duplicating a cube or trisecting an angle go back to the Greeks, and the problem of solving a cubic, quartic or quintic equation to the Renaissance. Many of the problems that are raised are of a concrete kind (and this, surely, is why it is so enjoyable) and yet the needs of the subject have led to substantial development in many branches of abstract algebra: in particular, in the theory of fields, the theory of groups, the theory of vector spaces and the theory of commutative rings.

In this book, Galois theory is treated as it should be, as a subject in its own right. Nevertheless, in the process, I have tried to show its relationship to various topics in abstract algebra: an understanding of the structures of abstract algebra helps give a shape to Galois theory and conversely Galois theory provides plenty of concrete examples which show the point of abstract theory.

The book comprises two unequal parts. In the first part, an account is given of the algebra that is needed for Galois theory. Much of this may well be familiar to the reader, but is included both for completeness and to introduce the terminology and notation that is used. Much of the algebra (groups, rings, fields and vector spaces) has general interest, and of course the development of Galois theory was responsible for the development of many algebraic ideas. We shall concentrate on presenting those algebraic ideas and results that are needed for Galois theory. For example, it is important to know that in the right circumstances, the factorization of polynomials with coefficients in a ring is essentially unique. Group theory plays a large part in Galois theory, but has developed into a huge subject. We shall concentrate on those parts, such as the theory of soluble groups, which are needed in Galois theory.

The second, more substantial, part is concerned with the theory of fields and with Galois theory, and contains the main material of the book; indeed, many readers may wish to start here and refer back to the first part as necessary. Of its nature, the theory develops an inexorable momentum. Nevertheless, there are many digressions (for example, concerning geometric constructions, finite fields and the solution of cubic and quartic equations): one of the pleasures of Galois theory is that there are many examples which illustrate and depend upon the general theory, but which also have an interest of their own. The high point of the book is of course the resolution of the problem of when a polynomial is solvable by radicals. I have, however, tried to emphasize (in the final chapter in particular) that this is not the end of the story: the resolution of the problem raises many new problems, and Galois theory is still a lively subject.

The last three chapters have a more abstract nature, and require the use of Zorn's lemma. In full generality, in the uncountable case, this depends upon the Axiom of Choice; this is discussed in the Appendix. Algebra is principally concerned with finite operations and relations, and is therefore largely concerned with finite or countable sets, and so these chapters have a rather hybrid quality.

Two hundred exercises are scattered through the text. It has been suggested to me that this is rather few: I think that anyone who honestly tries them all will disagree! In my opinion, textbook exercises are often too straightforward, but some of these exercises are quite hard. The successful solution of a challenging problem gives a much better understanding of the powers and limitations of the theory than any number of trivial ones. Remember that mathematics is not a spectator sport!