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## HOMOLOGICAL THEORY OF REPRESENTATIONS

Modern developments in representation theory rely heavily on homological methods. This book for advanced graduate students and researchers introduces these methods from their foundations up and discusses several landmark results that illustrate their power and beauty.

The categorical foundations include abelian and derived categories, with an emphasis on localisation, spectra, and purity. The representation theoretic focus is on module categories of Artin algebras, with discussions of the representation theory of finite groups and finite quivers. Also covered are Gorenstein and quasi-hereditary algebras, including Schur algebras, which model polynomial representations of general linear groups, and the Morita theory of derived categories via tilting objects. The final part is devoted to a systematic introduction to the theory of purity for locally finitely presented categories, covering pure-injectives, definable subcategories, and Ziegler spectra.

With its clear, detailed exposition of important topics in modern representation theory, many of which have been unavailable in one volume until now, this book deserves a place in every representation theorist's library.

**Henning Krause** is Professor of Mathematics at Bielefeld University. He works in the area of representation theory of finite dimensional algebras, with a particular interest in homological structures. His previous publications include the *Handbook of Tilting Theory* (Cambridge, 2007). Professor Krause is Fellow of the American Mathematical Society.

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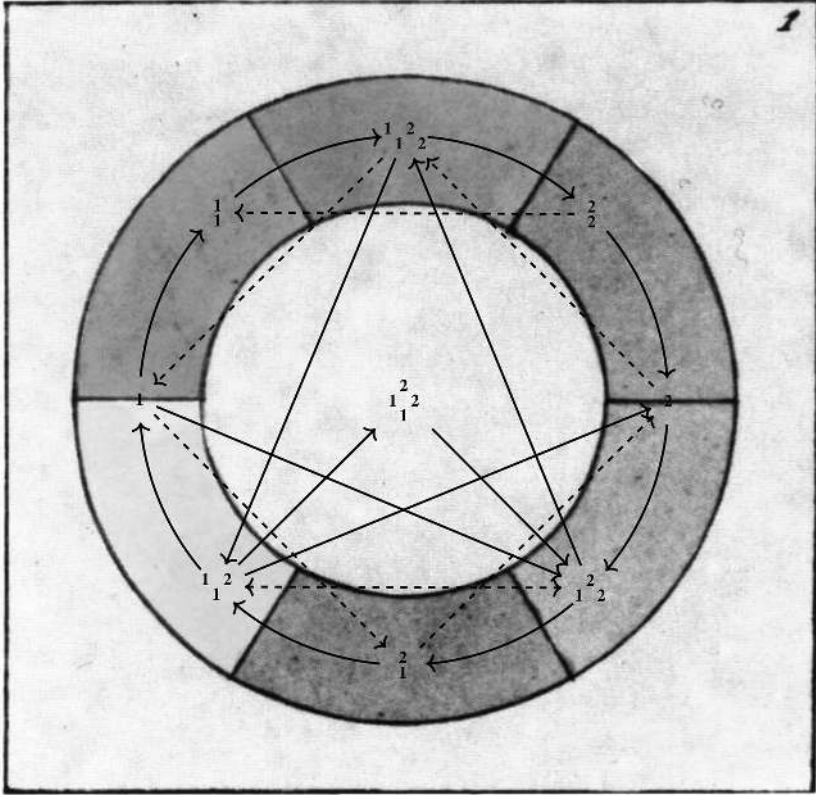
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# Homological Theory of Representations

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