HOMOLOGICAL THEORY OF REPRESENTATIONS

Modern developments in representation theory rely heavily on homological methods. This book for advanced graduate students and researchers introduces these methods from their foundations up and discusses several landmark results that illustrate their power and beauty.

The categorical foundations include abelian and derived categories, with an emphasis on localisation, spectra, and purity. The representation theoretic focus is on module categories of Artin algebras, with discussions of the representation theory of finite groups and finite quivers. Also covered are Gorenstein and quasi-hereditary algebras, including Schur algebras, which model polynomial representations of general linear groups, and the Morita theory of derived categories via tilting objects. The final part is devoted to a systematic introduction to the theory of purity for locally finitely presented categories, covering pure-injectives, definable subcategories, and Ziegler spectra.

With its clear, detailed exposition of important topics in modern representation theory, many of which have been unavailable in one volume until now, this book deserves a place in every representation theorist’s library.

Henning Krause is Professor of Mathematics at Bielefeld University. He works in the area of representation theory of finite dimensional algebras, with a particular interest in homological structures. His previous publications include the Handbook of Tilting Theory (Cambridge, 2007). Professor Krause is Fellow of the American Mathematical Society.
This illustration combines Goethe’s Farbkreis [J. W. von Goethe, Zur Farbenlehre, Erster Band, Nebst einem Hefte mit sechzehn Kupfertafeln, Tübingen, 1810] with the Auslander–Reiten quiver of a Gorenstein algebra of dimension one (Figure 6.1).
Homological Theory of Representations

HENNING KRAUSE
Universität Bielefeld, Germany
## Contents

<table>
<thead>
<tr>
<th>Section</th>
<th>Page</th>
</tr>
</thead>
<tbody>
<tr>
<td>Introduction</td>
<td>x</td>
</tr>
<tr>
<td>Conventions and Notations</td>
<td>xv</td>
</tr>
<tr>
<td>Glossary</td>
<td>xvii</td>
</tr>
<tr>
<td>Standard Functors and Isomorphisms</td>
<td>xxxiii</td>
</tr>
<tr>
<td><strong>PART ONE  ABELIAN AND DERIVED CATEGORIES</strong></td>
<td></td>
</tr>
<tr>
<td>1 Localisation</td>
<td>3</td>
</tr>
<tr>
<td>1.1 Localisation</td>
<td>3</td>
</tr>
<tr>
<td>1.2 Calculus of Fractions</td>
<td>10</td>
</tr>
<tr>
<td>Notes</td>
<td>12</td>
</tr>
<tr>
<td>2 Abelian Categories</td>
<td>14</td>
</tr>
<tr>
<td>2.1 Exact Categories</td>
<td>15</td>
</tr>
<tr>
<td>2.2 Localisation of Additive and Abelian Categories</td>
<td>28</td>
</tr>
<tr>
<td>2.3 Module Categories and Their Localisations</td>
<td>42</td>
</tr>
<tr>
<td>2.4 Commutative Noetherian Rings</td>
<td>47</td>
</tr>
<tr>
<td>2.5 Grothendieck Categories</td>
<td>55</td>
</tr>
<tr>
<td>Notes</td>
<td>70</td>
</tr>
<tr>
<td>3 Triangulated Categories</td>
<td>72</td>
</tr>
<tr>
<td>3.1 Triangulated Categories</td>
<td>73</td>
</tr>
<tr>
<td>3.2 Localisation of Triangulated Categories</td>
<td>77</td>
</tr>
<tr>
<td>3.3 Frobenius Categories</td>
<td>83</td>
</tr>
<tr>
<td>3.4 Brown Representability</td>
<td>89</td>
</tr>
<tr>
<td>Notes</td>
<td>100</td>
</tr>
<tr>
<td>4 Derived Categories</td>
<td>101</td>
</tr>
<tr>
<td>4.1 Derived Categories</td>
<td>102</td>
</tr>
<tr>
<td>4.2 Resolutions and Extensions</td>
<td>110</td>
</tr>
</tbody>
</table>

vii
Contents

4.3 Resolutions and Derived Functors 122
4.4 Examples of Derived Categories 133
Notes 144

5 Derived Categories of Representations 146
5.1 Examples Related to the Projective Line 146
5.2 Derived Categories of Finitely Presented Modules 164
Notes 172

PART TWO ORTHOGONAL DECOMPOSITIONS 173

6 Gorenstein Algebras, Approximations, Serre Duality 175
6.1 Approximations 176
6.2 Gorenstein Rings 179
6.3 Serre Duality 189
6.4 The Derived Nakayama Functor 195
6.5 Examples 203
Notes 205

7 Tilting in Exact Categories 207
7.1 Cotorsion Pairs 208
7.2 Tilting in Exact Categories 215
Notes 226

8 Polynomial Representations 228
8.1 Quasi-hereditary Algebras 231
8.2 Symmetric Tensors 241
8.3 Polynomial Representations 250
8.4 Cauchy Decompositions 264
8.5 Schur and Weyl Modules and Functors 272
8.6 Schur Algebras 284
Notes 291

PART THREE DERIVED EQUIVALENCES 295

9 Derived Equivalences 297
9.1 Differential Graded Algebras 298
9.2 Derived Equivalences 309
9.3 Finite Global Dimension 318
Notes 328
## Contents

10 Examples of Derived Equivalences 329
  10.1 Coherent Sheaves on Projective Space 329
  10.2 Koszul Duality 330
  10.3 The BGG Correspondence 332
  10.4 Koszul Duality for the Beilinson Algebra 333
  10.5 Weighted Projective Lines 334
  10.6 Gentle Algebras 336

PART FOUR  PURITY  339

11 Locally Finitely Presented Categories 341
  11.1 Locally Finitely Presented Categories 342
  11.2 Grothendieck Categories 356
  11.3 Gröbner Categories 367
  Notes 376

12 Purity  377
  12.1 Purity 378
  12.2 Definable Subcategories 384
  12.3 Indecomposable Pure-Injective Objects 392
  12.4 Pure-Injective Modules 400
  Notes 412

13 Endofiniteness  414
  13.1 Endofinite Objects and Subadditive Functions 415
  13.2 Endofinite Modules 426
  Notes 433

14 Krull–Gabriel Dimension  435
  14.1 The Krull–Gabriel Filtration 435
  14.2 Examples of Krull–Gabriel Filtrations 444
  Notes 456

References 457
Notation 469
Index 477