Introduction

This book is devoted to representations of associative algebras and their homological theory. There are two basic approaches. The passage from representations to chain complexes of representations leads to the study of *derived categories*. The other approach identifies representations with appropriate functors; this leads to the study of *functor categories*. We offer an introduction to both approaches and present results which illustrate their beauty and importance.

History. The appearance of the book *Homological Algebra* by Cartan and Eilenberg in 1956 established the subject [46]. In the following year Grothendieck published his seminal paper *Sur quelques points d'algèbre homologique* which inspired a whole generation [94]. Two students from the circle around Grothendieck then developed the foundations for the subject of this book. There is the thesis *Des catégories abéliennes* from 1960 by Peter Gabriel [79] and the thesis *Des catégories dérivées des catégories abéliennes* from 1967 by Jean-Louis Verdier [199]. Substantial parts of this book are devoted to explaining their work so that it can be applied to the study of representations.

Topics. We focus on representation theoretic results, most of which originate from the 1980s and early 1990s. A major part of the book is devoted to derived categories, and the notion of tilting plays an important role. Orthogonal decompositions provide another organisational principle for several results. The final part is about purity and involves the use of functor categories. The context for most results is the category of modules over a ring. When appropriate we work more generally with abelian categories, or we restrict to certain classes of modules or rings. For instance, of particular interest from the representation theory perspective are modules of finite length over Artin algebras.

The following is a list of topics and results which are treated in this book, beyond the foundational material discussed further below.

- *Gorenstein algebras.* The module category of a Gorenstein algebra admits an orthogonal decomposition into the subcategory of modules of finite projective dimension and the subcategory of Gorenstein projective modules. For Artin algebras the corresponding bounded derived category of modules of finite projective dimension and the stable category of Gorenstein projective modules admit Serre functors.
- *Tilting modules.* For Artin algebras there is a bijective correspondence between equivalence classes of tilting modules and covariantly finite coresolving subcategories.
- *Characteristic tilting modules.* For every quasi-hereditary algebra there is a canonical tilting module; its indecomposable direct summands are precisely the indecomposable modules which have a standard and a costandard filtration.
- *Schur algebras.* Polynomial representations of general linear groups identify with modules over Schur algebras. Every Schur algebra is quasi-hereditary and the characteristic tilting modules are given by tensor products of exterior powers.
- *Happel's theorem.* A tilting object of an exact category induces a triangle equivalence between its bounded derived category and the category of perfect complexes over the endomorphism algebra of the tilting object.
- *Happel's functor.* The bounded derived category of modules over an Artin algebra embeds into the stable category of graded modules over the corresponding trivial extension algebra, and equality holds if and only if the algebra has finite global dimension.
- *Rickard's theorem.* Two algebras have equivalent derived categories if and only if one admits a tilting complex with endomorphism algebra isomorphic to the other algebra.
- *Global dimension.* Tilting preserves finite global dimension. If the bounded derived category of an abelian category of finite global dimension admits a tilting object, then its endomorphism ring is of finite global dimension.
- *Gröbner categories.* Representations of the category of finite sets in some locally noetherian Grothendieck category form again a locally noetherian Grothendieck category. This is a vast generalisation of Hilbert's basis theorem and has several applications.
- *Definable subcategories.* The definable subcategories of a module category (that is, subcategories closed under filtered colimits, products and pure submodules) are in bijective correspondence to Ziegler closed subsets of indecomposable pure-injective modules.
- *Injective cohomology representations.* For a finite group, every injective module over its cohomology ring can be realised as the cohomology of

a representation. Such a representation is essentially unique and Σ-pure-injective; therefore it decomposes uniquely into indecomposable representations corresponding to homogeneous prime ideals of the cohomology ring.

– *Endofinite modules.* Modules of finite length over their endomorphism ring decompose uniquely into indecomposables, and the isomorphism classes of indecomposables are in bijective correspondence to irreducible subadditive functions on finitely presented modules.
– *Finite representation type.* A ring is of finite representation type if and only if every module is endofinite.
– *Krull–Gabriel filtrations.* Pure-injective objects are classified via Krull–Gabriel filtrations. Examples include modules over Dedekind domains, quasi-coherent sheaves on the projective line, and representations of the Kronecker quiver.

Foundations. Several chapters of this book are devoted to basic concepts and foundational results. Let us mention some of these topics.

– *Localisation.* Localisation is a process of adding formal inverses to an algebraic structure; it is used throughout the book. The localisation of additive categories amounts to annihilating appropriate subcategories. For abelian and triangulated categories the morphisms of a localised category can be described via a calculus of fractions.
– *Abelian categories.* Abelian categories generalise module categories. Of particular interest are Grothendieck categories, which are precisely the localisations of module categories. Objects in these categories admit injective envelopes; so one can do homological algebra.
– *Triangulated and derived categories.* The derived category of an abelian category provides the proper context for studying derived functors. An important ingredient is the construction of resolutions. Triangulated categories form the appropriate categorical framework. Useful tools include Verdier localisation and Brown representability for cohomological functors.
– *Locally finitely presented categories.* These are cocomplete additive categories such that every object is a filtered colimit of finitely presented objects; in fact they are determined by their full subcategories of finitely presented objects. For locally finitely presented Grothendieck categories there is a well-developed theory of injective objects. Pure-injective objects are studied via an embedding into a Grothendieck category that is locally finitely presented.

There are further fundamental concepts that appear throughout this work. We mention the notion of a finitely presented (or coherent) functor. In fact, functor categories play an important role, not only because each representation (of an algebra, a quiver, or a group) may be viewed as a functor. The basic idea is to identify an object X of an additive category with the corresponding representable functor $\mathrm{Hom}(-, X)$, often restricted to some appropriate generating subcategory. In this way categories of representations are presented as categories of functors. This idea goes back to Gabriel [79] and Auslander [7], but continues to be useful, also in the study of triangulated categories [150].

Another key concept that pervades this book is the notion of a spectrum. Indecomposable representations are often viewed as points of some space. The analogue in commutative algebra is the Zariski spectrum, but the spectrum of indecomposable injective objects of a Grothendieck category is the more general concept which is used throughout.

Prerequisites. The exposition is demanding in terms of the background and mathematical experience expected of the reader. We assume a basic knowledge of representation theory and homological algebra, including the appropriate categorical language.

Basic concepts and facts that are used throughout the book are arranged in a glossary which also serves to fix notation. Some topics from the glossary are explained in more detail in later chapters.

For unexplained terminology and further details, the following books are recommended: Cartan and Eilenberg [46], Mac Lane [141] (homological algebra), Schubert [183] (categories), Lam [136] and Stenström [197] (rings and modules).

Organisation. This book does not attempt to give a complete and systematic introduction to the homological theory of representations. It is rather motivated by a series of representation theoretic results (cf. the above list) for which we provide proper foundations and complete proofs. The choice of these results is based on personal taste and is by no means systematic.

The material is organised into 14 chapters and each is devoted to a particular topic. We have tried to keep the chapters as independent of each other as possible. This causes some repetition, but it will help the reader who is only interested in a particular topic.

Each chapter ends with notes and historical comments. This compensates for the fact that the body of the text gives no credits for definitions and theorems. We make no attempt to discuss the early history of the subject. For instance, the development of 'modern algebra' by Emmy Noether and her school inspired many concepts and results which are presented in this volume; for a detailed account we refer to Corry [55].

A final warning seems appropriate. We try to present concepts and results in their natural generality, even though readers may only be interested in some special cases. For instance, we treat derived categories of *exact categories*, and it is obvious that these are abelian in most applications. Or we study purity for *locally finitely presented categories*, despite the fact that module categories are the most interesting examples. Our motivation for generality is twofold. Concepts and arguments often become more transparent by identifying the ingredients that are essential. And we believe in potential applications beyond those which are obvious and well known.

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Conventions and Notations

Categories

We follow the von Neumann–Bernays–Gödel set theory and distinguish between *sets* and *classes*. All categories are assumed to be *locally small* in the sense that the objects form a class and for each pair of objects the morphisms between them form a set. When a category is abelian or exact, we assume in addition that for each pair of objects the extensions (in the sense of Yoneda) form a set.

We denote by Set the category of *sets* and by Ab the category of *abelian groups*. The *cardinality* of a set X is denoted by card X.

Morphisms are composed from right to left. For the composite $X \xrightarrow{\alpha} Y \xrightarrow{\beta} Z$ we write $\beta\alpha$.

Functors $\mathcal{C} \to \mathcal{D}$ are by convention covariant. Replacing one of the categories by its opposite category identifies contravariant functors $\mathcal{C} \to \mathcal{D}$ with covariant functors $\mathcal{C}^{\mathrm{op}} \to \mathcal{D}$ or $\mathcal{C} \to \mathcal{D}^{\mathrm{op}}$.

Rings and Modules

All rings are associative and have a unit.

For a ring Λ we consider the category Mod Λ of *right* Λ*-modules* but drop the adjective ‘right’. Left Λ-modules are identified with modules over the *opposite ring* Λ^{op}. The full subcategory of finitely presented Λ-modules is denoted by mod Λ, and proj Λ denotes the full subcategory of finitely generated projective Λ-modules.

When Λ and Γ are k-algebras over a commutative ring k, then Γ-Λ-*bimodules* ${}_\Gamma M_\Lambda$ are identified with modules over the algebra $\Gamma^{\mathrm{op}} \otimes_k \Lambda$.

Numbers

We denote by $\mathbb{Z}$ the set of integers and write

$$\mathbb{N} = \{0, 1, 2, 3, \dots\}$$

for the set of non-negative integers.

Glossary

Category. A *category* $\mathcal{C}$ is given by a class of *objects* $\operatorname{Ob}\mathcal{C}$ and a class of *morphisms* $\operatorname{Mor}\mathcal{C}$, together with an associative unital composition. For objects $X, Y \in \mathcal{C}$ let $\operatorname{Hom}_{\mathcal{C}}(X,Y)$ denote the set of morphisms $X \to Y$, and $\mathrm{id}_X \colon X \to X$ the identity morphism. We write $\operatorname{End}_{\mathcal{C}}(X)$ for the set of endomorphisms of X. Sometimes we simplify the notation and write $\operatorname{Hom}(X,Y)$ or $\mathcal{C}(X,Y)$. The composition is given by a map

$$\operatorname{Hom}_{\mathcal{C}}(Z,Y) \times \operatorname{Hom}_{\mathcal{C}}(X,Y) \longrightarrow \operatorname{Hom}_{\mathcal{C}}(X,Z), \quad (\psi,\phi) \mapsto \psi\phi$$

for each triple of objects $X, Y, Z \in \mathcal{C}$.

The category $\mathcal{C}$ is *small* if $\operatorname{Ob}\mathcal{C}$ is a set, and $\mathcal{C}$ is *essentially small* if the isomorphism classes of objects in $\mathcal{C}$ form a set. The *opposite category* of $\mathcal{C}$ is denoted by $\mathcal{C}^{\mathrm{op}}$.

Morphisms. A morphism $X \to Y$ in a category $\mathcal{C}$ is a *monomorphism* if the induced map $\operatorname{Hom}(C,X) \to \operatorname{Hom}(C,Y)$ is injective for all $C \in \mathcal{C}$ (notation: $X \rightarrowtail Y$), an *epimorphism* if the map $\operatorname{Hom}(Y,C) \to \operatorname{Hom}(X,C)$ is injective for all $C \in \mathcal{C}$ (notation: $X \twoheadrightarrow Y$), and an *isomorphism* if the map $\operatorname{Hom}(C,X) \to \operatorname{Hom}(C,Y)$ is bijective for all $C \in \mathcal{C}$ (notation: $X \xrightarrow{\sim} Y$).

The morphisms in $\mathcal{C}$ form the *category of morphisms* $\mathcal{C}^{\mathbf{2}}$; this identifies with the category of functors $\mathbf{2} \to \mathcal{C}$ where $\mathbf{2}$ denotes the category given by two objects which are connected by one morphism.

Functor. A *functor* $F \colon \mathcal{C} \to \mathcal{D}$ is given by a map on objects $\operatorname{Ob}\mathcal{C} \to \operatorname{Ob}\mathcal{D}$, together with maps on morphisms

$$F_{X,Y} \colon \operatorname{Hom}_{\mathcal{C}}(X,Y) \longrightarrow \operatorname{Hom}_{\mathcal{D}}(FX,FY)$$

for all objects $X, Y \in \mathcal{C}$, preserving the composition of morphisms. The identity functor is denoted by $\mathrm{id}_{\mathcal{C}} \colon \mathcal{C} \to \mathcal{C}$.

The functor F is *faithful* if all $F_{X,Y}$ are injective and *full* if all $F_{X,Y}$ are surjective. The notation $\mathcal{C} \rightarrowtail \mathcal{D}$ is used when F is fully faithful.

We write $\mathcal{H}om(\mathcal{C},\mathcal{D})$ or $\mathrm{Fun}(\mathcal{C},\mathcal{D})$ for the 'category' of functors $\mathcal{C} \to \mathcal{D}$. The morphisms between two functors are the natural transformations, but we do not require that they form a set.

Essential image. The *essential image* of a functor $F\colon \mathcal{C} \to \mathcal{D}$ is the full subcategory of $\mathcal{D}$ given by $\operatorname{Im} F = \{Y \in \mathcal{D} \mid Y \cong F(X) \text{ for some } X \in \mathcal{C}\}$. The functor F is *essentially surjective* if $\operatorname{Im} F = \mathcal{D}$.

Quotient functor. A *quotient functor* is a functor $F\colon \mathcal{C} \to \mathcal{D}$ such that for some class $S \subseteq \operatorname{Mor}\mathcal{C}$ of morphisms the functor F inverts all morphisms in S (so $F\phi$ is invertible for all $\phi \in S$) and every functor $F'\colon \mathcal{C} \to \mathcal{D}'$ factors uniquely through F provided that F' inverts all morphisms in S. In this case we set $\mathcal{C}[S^{-1}] = \mathcal{D}$ and the notation $\mathcal{C} \twoheadrightarrow \mathcal{D}$ is used for F.

Equivalence. A functor $F\colon \mathcal{C} \to \mathcal{D}$ is an *equivalence* if there is a functor $G\colon \mathcal{D} \to \mathcal{C}$, together with natural isomorphisms $\mathrm{id}_{\mathcal{C}} \xrightarrow{\sim} GF$ and $FG \xrightarrow{\sim} \mathrm{id}_{\mathcal{D}}$. An equivalent condition is that F is fully faithful and every object $Y \in \mathcal{D}$ is isomorphic to FX for some object $X \in \mathcal{C}$.[1] Notation: $\mathcal{C} \xrightarrow{\sim} \mathcal{D}$.

Adjoint. A pair (F,G) of functors $F\colon \mathcal{C} \to \mathcal{D}$ and $G\colon \mathcal{D} \to \mathcal{C}$ is *adjoint* if there are natural bijections:

$$\operatorname{Hom}_{\mathcal{D}}(FX,Y) \cong \operatorname{Hom}_{\mathcal{C}}(X,GY) \qquad (X \in \mathcal{C}, Y \in \mathcal{D}).$$

Then the notation $\mathcal{C} \rightleftarrows \mathcal{D}$ is used and there are two natural morphisms:

$$\eta_X\colon X \longrightarrow GF(X) \quad \text{(unit)} \qquad\qquad \varepsilon_Y\colon FG(Y) \longrightarrow Y \quad \text{(counit)}.$$

The composite

$$\operatorname{Hom}_{\mathcal{C}}(X',X) \xrightarrow{F} \operatorname{Hom}_{\mathcal{D}}(F(X'),F(X)) \xrightarrow{\sim} \operatorname{Hom}_{\mathcal{C}}(X',GF(X))$$

is given by composition with the unit η_X, and the composite

$$\operatorname{Hom}_{\mathcal{D}}(Y,Y') \xrightarrow{G} \operatorname{Hom}_{\mathcal{C}}(G(Y),G(Y')) \xrightarrow{\sim} \operatorname{Hom}_{\mathcal{D}}(FG(Y),Y')$$

is given by composition with the counit ε_Y. Moreover, the following conditions are equivalent.[2]

(1) The functor G is fully faithful.

[1] [197, Proposition IV.1.1]
[2] Proposition 1.1.3

(2) The counit ε_Y is an isomorphism for every object $Y \in \mathcal{D}$.
(3) The functor F is the composite $\mathcal{C} \twoheadrightarrow \mathcal{C}[S^{-1}] \xrightarrow{\sim} \mathcal{D}$ of a quotient functor for some class $S \subseteq \operatorname{Mor} \mathcal{C}$ and an equivalence.

This is expressed by the following diagram.

$$\mathcal{C} \underset{G}{\overset{F}{\rightleftarrows}} \mathcal{D}$$

Sometimes we denote by F_λ the left adjoint of F, and by F_ρ the right adjoint of F.

Localisation functor. A functor $L\colon \mathcal{C} \to \mathcal{C}$ is called a *localisation functor* if there exists a morphism $\eta\colon \operatorname{id}_{\mathcal{C}} \to L$ such that $L\eta\colon L \to L^2$ is an isomorphism and $L\eta = \eta L$.

Any localisation functor gives a pair (F, G) of adjoint functors such that F is (up to an equivalence) a quotient functor and G is fully faithful (by taking the inclusion $G\colon \mathcal{D} \to \mathcal{C}$ for $\mathcal{D} = \{X \in \mathcal{C} \mid \eta_X \text{ is invertible}\}$ and $FX = LX$ for every object $X \in \mathcal{C}$). Conversely, any pair (F, G) of adjoint functors such that F is a quotient functor or G is fully faithful gives a localisation functor (L, η) (by taking $L = GF$ and for η the unit).[3]

Limit and colimit. Let $\mathcal{C}$ be a category. A *diagram* of *type* $\mathcal{I}$ is a functor $\mathcal{I} \to \mathcal{C}$ where the category $\mathcal{I}$ is essentially small. Let $\mathcal{C}^{\mathcal{I}}$ denote the category of such diagrams. The *diagonal functor* $\Delta\colon \mathcal{C} \to \mathcal{C}^{\mathcal{I}}$ takes an object to the constant functor. For $F \in \mathcal{C}^{\mathcal{I}}$ the *limit* $\lim F$ (also written $\lim_{i \in \mathcal{I}} F(i)$) is given by a natural bijection

$$\operatorname{Hom}_{\mathcal{C}}(X, \lim F) \cong \operatorname{Hom}_{\mathcal{C}^{\mathcal{I}}}(\Delta X, F) \qquad (X \in \mathcal{C})$$

provided it exists in $\mathcal{C}$. Thus the limit is the right adjoint of the diagonal functor and the counit provides canonical morphisms $\lim F \to F(i)$ for all $i \in \mathcal{I}$. Analogously, the *colimit* $\operatorname{colim} F$ is given by

$$\operatorname{Hom}_{\mathcal{C}}(\operatorname{colim} F, X) \cong \operatorname{Hom}_{\mathcal{C}^{\mathcal{I}}}(F, \Delta X) \qquad (X \in \mathcal{C})$$

and comes with canonical morphisms $F(i) \to \operatorname{colim} F$ for all $i \in \mathcal{I}$.

Filtered category. A category $\mathcal{I}$ is *filtered* if

(Fil1) the category is non-empty,
(Fil2) given objects i, i' there is an object j with morphisms $i \to j \leftarrow i'$, and

[3] Proposition 1.1.5