Problems in Quantum Field Theory

This collection of problems in quantum field theory, accompanied by their complete solutions, aims to bridge the gap between learning the foundational principles and applying them practically. The carefully chosen problems cover a wide range of topics, starting from the foundations of quantum field theory and the traditional methods in perturbation theory, such as LSZ reduction formulas, Feynman diagrams and renormalization. Separate chapters are devoted to functional methods (bosonic and fermionic path integrals; worldline formalism), to non-Abelian gauge theories (Yang–Mills theory, quantum chromodynamics), to the novel techniques for calculating scattering amplitudes and to quantum field theory at finite temperature (including its formulation on the lattice, and extensions to systems out of equilibrium). The problems range from those dealing with QFT formalism itself to problems addressing specific questions of phenomenological relevance, and they span a broad range in difficulty, for graduate students taking their first or second course in QFT.

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With Fully-Worked Solutions

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To Kanako, Nathan and Simon.

Reality is that which, when you stop believing in it, doesn't go away.

PHILIP K. DICK

How to build a universe that doesn't fall apart two days later (1978)

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Preface

This project grew out as an extension to my previous *Quantum Field Theory* book (hereafter referred to as book I), in the form of an additional set of solved problems. The starting observation was that most textbooks (and mine is no exception) have a general inclination towards the exposition of the concepts, rather than the more practical aspects. This is of course quite understandable for such a vast subject, where one needs to absorb a quite large volume of concepts before becoming operational. The intended goal of the present book is to help fill the gap between theory and applications by providing a text almost exclusively geared towards actual practice (but occasionally new concepts are also introduced).

The set of problems included in this book cover most of the subjects treated in book I, and expose the reader to a broad variety of techniques. Occasionally, the same question is addressed by various methods, in order to shed light on it from different perspectives. The problems proposed in this set cover a broad range of difficulties and fall into several categories:

- More sophisticated and lengthier applications of the techniques and results of book I, that are too long to be treated as examples in a textbook
- Problems that extend book I towards new topics and concepts that are too specialized to fit reasonably in a textbook
- Classic results of quantum field theory (QFT) presented in a more modern fashion and with uniform notation, with the goal of making them more accessible than from a reading of the original literature
- More recent results, presented in a way that is accessible to the readers of book I; these are meant to be a bridge between the material of a typical textbook and contemporary research articles.

When deciding which problems to include in this collection, I made the deliberate choice to keep almost exclusively problems that can be worked out analytically "by hand" in a reasonable amount of time. Occasionally, straightforward but tedious calculations have been avoided by the use of a computer algebra system (for these problems, PYTHON notebooks will be provided as a separate online resource). This prejudice is of course an important limitation since these computer tools are a common aid in contemporary theoretical research, but my impression is that the didactical virtue of working out simpler problems by hand is higher.

Although this book has an obvious lineage with book I (e.g., the two books share the same notation), a significant effort has been made to ensure that it is self-contained and can be used on its own. Each chapter starts with a reminder of the important concepts and tools relevant for the problems of that chapter, but an important word of caution is in order here: these introductions are meant to be a concise refresher for a reader who has already studied the corresponding subjects, but they are not an appropriate source for learning a subject for the first time. This book is also self-contained by the inclusion of detailed solutions to all the problems. When necessary, a quick exposition of some relevant mathematical tools is included in the solutions, to avoid a detour via a mathematical textbook (this is intended to give plausibility to a given mathematical statement, not to provide a thorough and rigorous description of the underlying mathematics).

The intended readership of this book is of course primarily students who are in the process of learning quantum field theory, as well as their instructors. Roughly speaking, Chapters 1 and 2 cover the topics one would usually learn in a first QFT course. Chapter 3, on non-Abelian gauge theories, tends to be a bit more advanced and is often treated in a second course. Chapters 4 and 5 deal with more specialized subjects, respectively the newly developed tools for calculating scattering amplitudes, and aspects of quantum field theory that are on the border of many-body physics. Hopefully, more experienced readers will also find the book useful, both for the discussion of these more advanced topics and for a more modern treatment of some classic QFT results.

Acknowledgements

This book would not exist without the many questions from students to whom I had the pleasure of teaching quantum field theory during the past five years at École Polytechnique. Some of these questions were indicative of a demand to go beyond the formal developments that constitute the main body of the course. Occasionally, these questions would touch on subtle and not often discussed points of QFT. Many problems in this book are the result of my attempts to clarify these points, for myself first, before I could comfortably provide an answer to my students.

This book also owes a lot to Vince Higgs at Cambridge University Press, for his supportive reception of a preliminary draft and for his precious advice regarding how to improve it. I would like to extend these thanks to all the support staff at Cambridge University Press, who helped me a lot in the more technical aspects of making this book, with a special mention to Elle Ferns and Henry Cockburn, and to the anonymous referees whose feedback was very helpful. Last but not least, John King, who went through the painstaking task of copy-editing the manuscript, provided me with immensely valuable feedback that considerably improved its readability.

Finally, I would like to address my warmest thanks to my wife, Kanako, for her constant encouragement, support and patience. In normal times, writing a book is an activity which is already quite prone to interfering with family life. The covid pandemic, by forcing me to work from home most of the time during the preparation of this book, exacerbated this by blurring even more the boundary between work and leisure.

Notation and Conventions

We list here some notation and conventions that are used throughout this book:

- $c = 1$: length = time, energy = momentum = mass
- $\hbar = 1$: momentum = wavenumber, energy = frequency
- $p^\mu \equiv (p^0, \mathbf{p})$: 4-momentum; $\mathbf{p}$: three-dimensional spatial components
- $E_\mathbf{p} \equiv \sqrt{\mathbf{p}^2 + m^2}$: positive on-shell energy of a particle of momentum $\mathbf{p}$ and mass m
- $\mathcal{L}$: Lagrangian density (more rarely, the Lagrangian itself)
- $\mathcal{H}$: Hamiltonian; $\mathcal{S}$: action
- $\mu, \nu, \rho, \sigma, \ldots$ (more rarely $\alpha, \beta, \gamma, \ldots$): Lorentz indices in Minkowski space
- $i, j, k, l, \ldots$: Lorentz indices in Euclidean space
- $a, b, c, d, \ldots$: group indices in the adjoint representation
- $i, j, k, l, \ldots$: group indices in the fundamental representation
- $\epsilon_{\mu\nu\rho\cdots}$: Levi–Civita symbol. In situations where it makes sense to raise or lower the indices (e.g., for Lorentz indices in Minkowski space), the normalization convention is with lowered indices, $\epsilon_{012\cdots} = +1$
- The normalization of creation and annihilation operators is defined so that the canonical commutation relation reads $[a_\mathbf{p}, a^\dagger_\mathbf{q}] = (2\pi)^3 2E_\mathbf{p}\delta(\mathbf{p} - \mathbf{q})$ (except in Problems 80 and 81, where it is convenient to omit the factor $2E_\mathbf{p}$). Their dimension is $(\text{mass})^{-1}$.