Introduction

In this monograph, we introduce the reader to the connection between topological dynamical systems and dimension groups. Let us first explain briefly each of these terms.

A topological dynamical system is a pair (X, T) of a compact metric space X and a continuous map T from X to itself. The system is minimal if the orbit of every point of X is dense. We will be mainly interested in minimal systems. We will often consider the case where X is a closed subset of the set of $A^{\mathbb{Z}}$ of infinite sequences over a finite alphabet A, and T is the shift on $A^{\mathbb{Z}}$. When X is, moreover, invariant by T, we obtain a topological dynamical system called a shift space.

A dimension group is an ordered abelian group having some additional specific properties. To every minimal shift space (and more generally to every minimal Cantor system), we will associate a dimension group in such a way that isomorphic systems have isomorphic dimension groups.

One of the main objects of this book is to describe various methods to compute these dimension groups. In this way, we will be able to distinguish topological dynamical systems, which can appear in many different forms, by comparing their dimension groups, which are easier to handle.

As a motivating example, consider the Fibonacci sequence, which is the sequence

$$x = abaababa \cdots$$

obtained by iterating indefinitely the substitution $a \mapsto ab$, $b \mapsto a$ starting with a. The sequence of iterates

$$\begin{array}{l} a \\ ab \\ aba \\ abaab \\ \cdots \end{array}$$

converges (in an obvious sense) to x. The shift space X formed of the sequences $y \in \{a, b\}^{\mathbb{Z}}$ with all its blocks appearing in x is a shift space called the Fibonacci shift. We will see that its dimension group is the discrete subgroup of $\mathbb{R}$ formed of the $x + y\alpha$ with $x, y \in \mathbb{Z}$ and where $\alpha = (1 + \sqrt{5})/2$ is the golden mean. We shall see how this is related to the fact that there is a unique invariant probability measure on X that is such that the probability of the set of sequences $y = (y_n)_{n\in\mathbb{Z}}$ such that $y_0 = a$ is $\alpha - 1$.

Dimension groups were first associated with a family of associative algebras called approximately finite, or AF-algebras. These algebras are themselves a class of C^*-algebras that are direct limits of finite dimensional algebras and were introduced by Ola Bratteli (1972). The algebra is built from a special kind of graph called a *Bratteli diagram*.

Dimension groups were introduced by George Elliott (1976) as a tool for classifying AF-algebras and he proved that the dimension group (together with an additional information called the scale) provides a complete algebraic invariant for these algebras.

The connection of these ideas with dynamical systems was first done by Wolfgang Krieger (1977) (see also Krieger (1980a)) who defined a dimension group for every shift of finite type. The link with Bratteli diagrams was done by Anatol Vershik (1982) who used a lexicographic order on paths of the Bratteli diagrams to define a topological dynamical system on the set of infinite paths of the graph. Later, Richard Herman, Ian Putnam and Christian Skau showed that every minimal system on a Cantor space is isomorphic to such system. As a consequence, a dimension group is attached to any minimal Cantor system and subsequent work by Thierry Giordano, Ian Putnam and Christian Skau (1995) showed that this group is related to the orbit structure of the system.

In this expository presentation, written after the unpublished notes by Bernard Host (1995) (see also Host (2000)), we present the basic elements of this theory, insisting on the computational and algorithmic aspects allowing one to effectively compute the dimension groups. The computation applies in particular to the case of substitution shifts, explicitly presented previously in Durand et al. (1999), in relation with Forrest (1997).

In the first chapter (Chapter 1) we present the basic notions of topological dynamical systems. Although such systems can be defined using a group or semigroup action, we restrict our attention to systems on which acts the group $\mathbb{Z}$ or the semigroup $\mathbb{N}$. We define recurrent systems and minimal dynamical systems (Section 1.1). Next, we introduce in Section 1.2 shift spaces, which are the basic systems we are interested in. We define return words and higher block shifts. In Section 1.4, we introduce substitution shifts. We define the notion of recognizable morphism and we prove the Theorem of Mossé (Theorem 1.4.35) asserting that any aperiodic primitive morphism is recognizable.

In the second chapter (Chapter 2), we shift to an algebraic and combinatorial environment. We first introduce, in Section 2.1, ordered groups (considering only abelian groups). We define several notions, as that of order unit and order ideal. We also define a simple ordered group as one with no nontrivial ideals. In Section 2.3 we define direct limits of ordered groups and we give examples of the computation of these ordered groups. In the last part of this section (Section 2.4), we finally define dimension groups. These groups are defined as direct limits of groups $\mathbb{Z}^n$ with the usual ordering. We prove the abstract characterization by Effros, Handelman and Shen Effros et al. (1980) using the property of Riesz interpolation.

In Chapter 3, we come to notions of cohomology defined in a Cantor system. We first introduce the notion of coboundary (Section 3.1) and prove in Section 3.2 the Gottshalk–Hedlund Theorem (Proposition 3.2.5) characterizing the continuous functions on a Cantor set that are coboundaries. We next define the ordered cohomology group $K^0(X, T)$ of a recurrent system (X, T) as the quotient of the group of integer valued continuous functions on X by the subgroup formed by coboundaries. In the next two sections (Sections 3.6 and 3.7), we consider the effect on the ordered cohomology group of applying a factor map or taking the system induced on a clopen set. In the second part of this chapter, beginning with Section 3.8, we define invariant probability measures on a Cantor system and recall that a substitutive shift defined by a primitive substitution has a unique invariant probability measure. We indicate a method to compute this measure. We show in Section 3.9 that there is a close connection between the cohomology group and the cone of invariant measures (Proposition 3.9.3). We use this connection to give a description of the dimension groups of Sturmian shifts (Theorem 3.9.3).

In Chapter 4, we introduce the fundamental tool of partitions in towers, or Kakutani–Rokhlin partitions. We prove the theorem of Herman, Putnam and Skau, which shows that any minimal Cantor system can be represented as the limit of a sequence of partitions in towers (Theorem 4.1.6). In Chapter 4 we come back to partition in towers. We first show how to associate an ordered group to a partition in towers (Section 4.2). Next, in Section 4.3, we use a sequence of partitions in towers to prove that the group $K^0(X, T)$ is, for any minimal dynamical system (X, T), a simple dimension group (Theorem 4.3.4). In the next sections, we present explicit methods to compute the dimension group of a minimal shift space. In Section 4.4, we use return words and in Section 4.5, we use Rauzy graphs. Finally, in Section 4.6, we show how to compute the dimension group of a substitutive shift, as exposed in Durand et al. (1999).

We introduce Bratteli diagrams in Chapter 5. We define the telescoping of a diagram. We define the dimension group of a Bratteli diagram and prove that it is a complete invariant for telescoping equivalence (Theorem 5.1.5). We next introduce ordered Bratteli diagrams and show that one may associate a dynamical system to every properly ordered Bratteli diagram. We prove the Bratteli–Vershik Model Theorem (Theorem 5.3.3) showing the completeness of the model for minimal Cantor systems. We next prove the Strong Orbit Equivalence Theorem (Theorem 5.5.1) showing that dimension groups are a complete invariant for strong orbit equivalence. We state (without proof) the related Orbit Equivalence Theorem (Theorem 5.5.3).

In Chapter 6, we focus on substitution shifts and their representations. We begin by considering odometers, which have BV-representations close to substitution shifts. We characterize, as a main result, the family of BV-systems associated with stationary Bratteli diagrams as the disjoint union of stationary odometers and substitution minimal systems (Theorem 6.2.1). We develop next the description of linearly recurrent shifts, which are characterized by their BV-representation (Theorem 6.3.5). We introduce in Section 6.4 the notion of an $\mathcal{S}$-adic representation. The main result is an explicit description of the dimension group of a unimodular $\mathcal{S}$-adic shift (Theorem 6.5.4). In the last section (Section 6.6), we consider the family of substitutive shifts, a natural generalization of substitution shifts. The main result is a characterization by a finiteness condition of substitutive sequences (Theorem 6.6.1).

Chapter 7 describes the class of dendric shifts, defined by a restrictive condition on the possible extensions of a word. This class is a simultaneous generalizations of several other classes of interest, such as Sturmian shifts or interval exchange shifts (introduced in the next chapter). The main result is the Return Theorem (Theorem 7.1.15), which states that the set of return words in a minimal dendric shift is a basis of the free GP on the alphabet. We use this result to describe the $\mathcal{S}$-adic representation of dendric shifts and show that it can be defined using elementary automorphisms of the free group (Theorem 7.1.40). We illustrate these results by considering the class of Sturmian shifts (Section 7.2). The last part of the chapter is devoted to specular shifts, a class of eventually dendric shifts that plays a role in the next chapter, when we introduce linear involutions. The main result is a description of the dimension group of a specular shift (Theorem 7.3.40).

In Chapter 8, we introduce the notion of interval exchange transformation. We prove Keane's Theorem characterizing minimal interval exchanges (Theorem 8.1.2). We develop the notion of Rauzy induction and characterize the subintervals reached by iterating the transformation (Theorem 8.1.25). We generalize Rauzy induction to a two-sided version and characterize the intervals reached by this more general transformation (Theorem 8.2.2). We link these transformations with automorphisms of the free group (Theorem 8.2.14). We also relate these results with the theorem of Boshernizan and Carroll giving a finiteness condition on the systems induced by an interval exchange when the lengths of the intervals belong to a quadratic field (Theorem 8.3.2). In the last section (Section 8.4) we define linear involutions and show that the natural coding of a linear involution without connections is a specular shift (Theorem 8.4.9).

In the last chapter (Chapter 9) we give a brief introduction to the link between Bratteli diagrams and the vast subject of C^*-algebras. We define approximately finite algebras and show their relation to Bratteli diagrams. We relate simple Bratteli diagrams and simple AF algebras (Theorem 9.3.12). We prove Elliott's Theorem showing that AF algebras are characterized by their dimension groups (Theorem 9.3.21).

A point useful to mention is that each chapter ends with exercises that can be either illustrations of the results or proofs of some results stated in the chapter, or additional results. For each of them, a solution is provided. The style of writing for the solutions is often more concise than for a proof in the main text but is in general a full proof.

After the exercises, a section of notes concludes each chapter, giving the bibliographic references and also pointing to further results.

The book ends with a series of appendices. The first one (Appendix A) gives the solutions of the exercises proposed in the previous chapters.

A second appendix (Appendix B) is a guide to be used as a reference for notions from several domains of mathematics used in this book. There are also three appendices of a special kind. The first one (Appendix C) is a summary of the various systems (or classes of systems) discussed in the chapters. Next, Appendix D lists the many equivalent definitions of Sturmian shifts (we hope to be thanked by the readers for this). Finally, Appendix E gives a list of open problems in this field.

The book is written in such a way that it should be readable by a graduate student in mathematics or computer science. As a general rule (with a few exceptions, and notably in Chapter 9), complete proofs are given. Some chapters can be read independently of the others, although most of them rely on the introductory chapter (Chapter 1). It seems impossible to cover all chapters in one course, but a selection can be made, resulting in a significant content. One of the authors has recently taught with success the content of Chapters 1, 2 and 3.

Acknowledgments This book started as notes following a seminar held in Marne-la-Vallée in June 2016 and gathering Marie-Pierre Béal, Valérie Berthé, Francesco Dolce, Pavel Heller, Revekka Kyriakoglou, Julien Leroy, Dominique Perrin and Giuseppina Rindone. The initial version followed closely the notes of Bernard Host (1995), trying to develop more explicitly some arguments. We have progressively enlarged it to its present version. We are deeply grateful to Bernard Host for letting us use his manuscript and encouraging us to continue to reach a more complete account of the field. We wish to thank Mike Boyle, Kenneth Davidson, Brian Marcus and Ian Putnam for fruitful discussions during the preparation of this text. We are also grateful to Paulina Cecchi, Francesco Dolce, Pavel Heller, Maryam Hosseini, Amir Khodayan Karim, Revekka Kyriakoglou, Sébastien Labbé, Christophe Reutenauer, Gwénaël Richomme and Wolfgang Steiner for reading the manuscript and finding many errors. Special thanks are due to Soren Eilers for reading closely Chapter 9 and to Christian Choffrut for his careful reading of most chapters. The manuscript was typeset using LaTeX for the text and Ti*k*Z for the figures. We acknowledge the use of many macros due to Jean Berstel.