Hardy Martingales

This book presents the probabilistic methods around Hardy martingales for an audience interested in applications to complex, harmonic, and functional analysis. Building on the work of Bourgain, Garling, Jones, Maurey, Pisier, and Varopoulos, it discusses in detail those martingale spaces that reflect characteristic qualities of complex analytic functions. Its particular themes are holomorphic random variables on Wiener space, and Hardy martingales on the infinite torus product, and numerous deep applications to the geometry and classification of complex Banach spaces, e.g., the SL^{∞} estimates for Doob's projection operator, the embedding of L^1 into L^1/H^1, the isomorphic classification theorem for the polydisk algebras, or the real variables characterization of Banach spaces with the analytic Radon Nikodym property. Due to the inclusion of key background material on stochastic analysis and Banach space theory, it's suitable for a wide spectrum of researchers and graduate students working in classical and functional analysis.

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Hardy Martingales

Stochastic Holomorphy, L^1-Embeddings, and Isomorphic Invariants

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Preface

In this book we present probabilistic methods developed for applications to complex and functional analysis. We will study, in depth, spaces of martingales that reflect characteristic qualities of holomorphic functions; specifically,

(i) the space of integrable Hardy martingales $H^1(\mathbb{T}^{\mathbb{N}})$ defined by restrictions on the support of their Fourier coefficients;
(ii) the space of holomorphic random variables $H^1(\Omega)$ on Wiener space, characterized by their Itô-integral representation.

Stochastic Holomorphy

The interplay between (nonconstant) holomorphic functions f and complex Brownian motion (z_t) goes back to the work of Paul Lévy who noted that $f(z_t)$ is the path of a complex Brownian motion (Lévy, 1948). More precisely, $f(z_t)$ is distributionally indistinguishable from $z_{\beta(t)}$ where

$$\beta(t) = \int_0^t |f'(z_s)|^2 ds.$$

Itô and McKean (1965) presented a proof of Picard's theorem (asserting that a nonconstant analytic function on $\mathbb{C}$ omits at most one value) by applying Lévy's result to the universal covering map onto $\mathbb{C}\backslash\{0, 1\}$.

Being more specific, we let $f : \mathbb{D} \to \mathbb{C}$ be analytic and bounded where $\mathbb{D} = \{z \in \mathbb{C} : |z| < 1\}$. We let τ denote the exit time of (z_t) from $\mathbb{D}$. The process $(f(z_t) : t < \tau)$ may be expanded by Itô integrals,

$$f(z_t) = \int_0^t f'(z_s)dz_s, \qquad t < \tau, \tag{0.0.1}$$

and hence forms a Brownian martingale. Doob (1953) proved martingale convergence theorems, showing that $\lim_{t\to\tau} f(z_t)$ exists almost surely, and developed the tools (conditioned Brownian motion) by which martingale convergence is transformed into radial limits such that

$$\lim_{r\to 1} f(r\zeta) \quad \text{exists for almost every} \quad \zeta \in \mathbb{T},$$

where $\mathbb{T} = \{z \in \mathbb{C} : |z| = 1\}$. Thus Fatou's theorem and Privalov's theorem were among the first results in complex analysis obtained by stochastic methods. Many years later Burkholder et al. (1971) showed that

$$\mathbb{E}|f(z_\tau)| \le C\mathbb{E}\sup_{s<\tau} |\Re f(z_s)|,$$

which in turn, by Doob's conditioned Brownian motion, gives rise to the real variable characterization of the Hardy space $H^1(\mathbb{T})$.

The Itô integral representation (0.0.1) provides the model for an intrinsically *stochastic* concept of *holomorphy*. We say that $F_t : \Omega \to \mathbb{C}$ is a holomorphic martingale on Wiener space $(\Omega, (\mathcal{F}_t), \mathbb{P})$ if there exists a complex-valued, $(\mathcal{F}_t)$ adapted process (X_t) satisfying

$$F_t = F_0 + \int_0^t X_s dz_s.$$

A holomorphic random variable is just the limit $F = \lim_{t\to\infty} F_t$ of an equiintegrable holomorphic martingale.

In applications to analysis, it is important that holomorphic martingales form a linear space and that they are stable under stopping times, pointwise multiplication, and composition with analytic functions. Varopoulos (1980, 1981) found a key to profound probabilistic methods in complex analysis by analyzing the action of Doob's conditional expectation operator,

$$N(F)(e^{i\theta}) = \mathbb{E}\left(F|z_\tau = e^{i\theta}\right),$$

on a holomorphic random variable F. It consists in linking the space of p-integrable holomorphic random variables $H^p(\Omega)$ to the corresponding Hardy space $H^p(\mathbb{T})$ by means of the commutative diagram,

$$\begin{array}{ccc} H^p(\mathbb{T}) & \xrightarrow{\quad \mathrm{Id} \quad} & H^p(\mathbb{T}) \\ & {\scriptstyle M}\searrow \quad \nearrow{\scriptstyle N} & \\ & H^p(\Omega) & \end{array},$$

where $Mf = f(z_\tau)$. Chapter 1 develops central ideas of stochastic holomorphy together with their applications to the Marcinkiewicz decomposition and to complex convexity estimates for $H^p(\mathbb{T})$ and for the Banach space $SL^\infty(\mathbb{T})$ of harmonic functions with uniformly bounded square functions.

Hardy Martingales and Banach Spaces

Let X be a complex Banach space. We turn now to vector-valued martingales on the infinite torus product $(\mathbb{T}^{\mathbb{N}}, (\mathcal{F}_k), \mathbb{P})$, where $\mathbb{P}$ is the Haar probability measure, and $(\mathcal{F}_k)$ the canonical filtration on $\mathbb{T}^{\mathbb{N}}$. Following Garling (1988), we say that an $(\mathcal{F}_k)$ martingale $(F_k : \mathbb{T}^{\mathbb{N}} \to X)$ is a *Hardy martingale* if conditioned on $\mathcal{F}_{k-1}$,

$$F_k - F_{k-1} \text{ defines an element in } H_0^1(\mathbb{T}, X).$$

This book is on Hardy martingales, their roots in the fields of stochastic, complex, and harmonic analysis, and their numerous, deep applications to the geometry and classification of complex Banach spaces.

We begin our review with the role vector-valued Hardy martingales play in the general study of Banach spaces and their isomorphic invariants. We single out the analytic Radon–Nikodym property (aRNP) of a complex Banach space X. A vector-valued analytic function $f : \mathbb{D} \to X$ may be defined simply by demanding that $x^*(f) : \mathbb{D} \to \mathbb{C}$ be analytic for each x^* in the dual space X^*. This condition implies the validity of the Cauchy integral formula for f, and hence its expansion into norm convergent power series with coefficients in X. It was observed very early on that the validity of Fatou's theorem is no longer assured, for any Banach space X. If for every bounded analytic $f : \mathbb{D} \to X$,

$$\lim_{r \to 1} f(r\zeta) \quad \text{exists in } X \text{ for almost every} \quad \zeta \in \mathbb{T},$$

then we say that X satisfies the aRNP. Heins (1969), for instance, proved that the aRNP holds true for the Banach spaces $\ell^p, 1 \leq p < \infty$, but not for c_0. Bukhvalov and Danilevich (1982) showed that it is satisfied for L^1, and, more generally, for any Banach lattice not containing c_0.

From the point of view of Banach space theory, such a phenomenon requires clarification. Considerable effort has gone into the determination of the geometric, probabilistic, and real variable characterization of those Banach spaces satisfying the aRNP. In this book, we present definitive characterizations of aRNP due to Ghoussoub, Lindenstrauss, and Maurey (Ghoussoub et al. [1989]; Ghoussoub and Maurey [1989]). While their conditions are expressed in purely analytic terms (the existence of small pluri-subharmonic slices and the pluri-subharmonic variational principle), the method of proof is a probabilistic one, exploiting vector-valued Hardy martingales on the infinite torus product.

Having pluri-subharmonic characterizations at hand, Ghoussoub, Maurey and Schachermayer were able to prove that J_*T, the predual of the James tree space, satisfies the aRNP (Ghoussoub and Maurey, 1989; Ghoussoub et al., 1989).

Above, we considered Hardy martingales and radial limits of analytic functions ranging in a general Banach space X. We now turn to describing the significance of Hardy martingales in the analysis of concrete Banach spaces of analytic functions. The following theorems were instrumental in establishing the importance of Hardy martingales in problems of hard analysis.

(i) Maurey (1980) started the proof that $H^1(\mathbb{T})$ is isomorphic to dyadic-H^1 by discretizing the process $(f(z_t)\colon t < \tau)$, with $f \in H^1(\mathbb{T})$. The result of the discretization is a Hardy martingale (F_k) with uniformly bounded increments satisfying

$$\mathbb{E}(\sup |F_n|) \leq C \sup_n \mathbb{E}|F_n| \quad \text{and} \quad |\Delta F_n| \leq \epsilon \int_{\mathbb{T}} |f| dm.$$

Maurey's discretization yields that any Hardy martingale may be regarded a subsequence of a Hardy martingale *with predictable increments* and almost identical L^1-norm.

(ii) The L^1 three-space problem asks if, for any subspace X of L^1, either X or L^1/X contain a subspace isomorphic to L^1. Pełczyński pointed out the importance of the case $X = H^1(\mathbb{T})$, which was resolved by Bourgain (1983a) in terms of Hardy martingales on the infinite torus product.

(iii) Bourgain's theorem (Bourgain, 1983a) asserts that L^1 is isomorphic to a subspace of L^1/H^1. In the converse direction, we have Pisier's (Davis et al., 1984) result that L^1/H^1 is not finitely represented in L^1. Pisier exhibited a uniformly bounded Hardy martingale $(G_n\colon \mathbb{T}^{\mathbb{N}} \to L^1/H^1)$ for which

$$\mathbb{E}\|\Delta G_n\|_{L^1/H^1_0} \geq 1,$$

for $n \in \mathbb{N}$, sharply contrasting with martingale cotype 2 estimates valid for Hardy martingales ranging in L^1. Since (G_n) is a sequence of Riesz products on $\mathbb{T}^{\mathbb{N}}$, an important link is hereby established between Hardy martingales and harmonic analysis on compact abelian groups.

(iv) Bourgain (1984b), connecting Pisier's analysis of vector-valued Riesz products and cotype 2 estimates for $A^*(\mathbb{T})$, showed that $A^*(\mathbb{T} \times \mathbb{T})$ is not finitely represented in $A^*(\mathbb{T})$. Multi-indexed Hardy martingales play a central role in the ensuing isomorphic classification theorem for polydisk algebras.

(v) Garling (1991) obtained the unconditional convergence of martingales by verifying it directly for Hardy martingales using Yudin's Lemma, and showing that the general result follows from the special case and the boundedness of the Riesz projection. Thus Garling's study determines the position of Hardy martingales within the field of harmonic analyis.

This book contains detailed proofs of the theorems listed above and provides a thorough investigation of complex Banach spaces vis-a-vis Hardy martingales. We investigate their convergence (aRNP), unconditional convergence (aUMD), the notions of Hardy martingale cotype and complex uniform convexity.

In writing the book, I intended to present elements of stochastic analysis and Banach space theory to the classical complex analyst – at the same time, I wanted the book to be useful for the functional analyst interested in Banach spaces of analytic functions and/or probability.

Contents

Chapter 1 presents elements of stochastic analysis, emphasizing Itô calculus, Doob's embedding and projection, holomorphic martingales, stopping time decompositions, and SL^∞ estimates by means of conditioned Brownian motion.

Chapter 2 gives an overview of Hardy martingales, highlighting examples originating in harmonic analysis on $\mathbb{T}^{\mathbb{N}}$, and complex Brownian martingales. We obtain central norm estimates for projection operators onto Hardy martingales and spaces of homogeneous polynomials, prove L^1 estimates for the martingale square function, and, by nonlinear teleskoping, we prove the Davis and Garsia inequality for Hardy martingales.

Chapter 3 is devoted to proving that L^1 embeds into $L^1(\mathbb{T})/H^1_0(\mathbb{T})$. The heart of the matter is the corresponding result for the quotient space formed by martingales on the infinite torus product

$$L^1(\mathbb{T}^{\mathbb{N}})/H^1_0(\mathbb{T}^{\mathbb{N}}).$$

Once this is established, the transfer methods developed by Bourgain, Bonami, and Meyer yield the embedding of L^1 into $L^1(\mathbb{T})/H^1_0(\mathbb{T})$.

Chapter 4 begins with Kwapien's representation of operators on L^0 and some of its consequences. We work through the examples of L^1-subspaces (other than H^1) for which a positive solution to the L^1 three-space problem is known. These are the reflexive subspaces of L^1, subspaces isomorphic to ℓ^1, and the complemented subspaces. We present Talagrand's space $X \subset L^1$ and prove that neither X nor L^1/X contain a copy of L^1, contrasting the Johnson–Maurey–Schechtman construction of a normalized weakly null sequence (φ_i) in L^1 that fails to contain an unconditional subsequence.

Chapter 5 presents the proof of the dimension conjecture for polydisk algebras and investigates basic isomorphic invariants of complex Banach spaces induced by Hardy martingales: the analytic Radon–Nikodym property, the analytic UMD property, and Hardy martingale cotype.

Chapter 6 continues the analysis of the projection operators introduced in Chapter 2. We prove Rosenthal's L^p theorem and utilize it to show that these operators are bounded on $L^p(Y)$ where Y is a reflexive subspace of L^1 and $1 < p < \infty$.

Chapter 7 is a case by case study of the isomorphic invariants associated with Hardy martingales, covering L^1 quotients by reflexive spaces, the trace class, iterated $L^p(L^q)$-spaces, the James tree space, and L^1/H^1. The following results are presented:

(i) L^1 quotients by reflexive spaces are of cotype 2 (Kislyakov, Pisier) and satisfy aUMD (Bourgain and Davis).
(ii) The trace class S^1 satisfies Hardy martingale cotype-2 estimates and fails aUMD (Haagerup and Pisier).
(iii) The iterated $L^p(L^q)$-spaces are super-reflexive Banach lattices, with an unconditional basis and if $p \neq q$, fail aUMD (Qiu).
(iv) The predual of the James tree space satisfies the aRNP (Ghoussoub, Maurey, and Schachermayer).
(v) L^1/H^1 is of cotype 2 (Bourgain). Fails the aRNP.

Prerequisites. For the most part our prerequisites are contained in the books by Dieudonné (1968a,b) and Rudin (1974, 1991) and in the first chapters of Durrett (1984) and Wojtaszczyk (1991). There are instances, however, mainly in Chapter 7, where we rely on more specialized sources.

Acknowledgments

While writing this book, I benefited from having access to several unpublished sources dating back to the 1980s. These are the notes by Pallares and Kouba (1986–1987), recording the seminar talks at Groupe de Travail sur les Espaces de Banach directed by Maurey and Pisier; the lecture notes by Pisier entitled "H^p-spaces of Banach space valued functions" (1991, unpublished); Garling's exposition of holomorphic martingales (Garling, D. J. H. $L^1(\Omega)/H^1(\Omega)$ H is a Grothendieck space of cotype 2. pages 1–18, 1992, unpublished.), and detailed seminar notes by Schmuckenschläger on complex uniform convexity (1993/4, unpublished).

I presented several parts of Chapters 1, 2, 3, and 5 at Texas A&M University during SUMIRFAS meetings, at the Mathematical Institute of the Polish Academy of Sciences (Warszawa and Gdansk), and at Yale University. It is my pleasure to thank S. Dilworth, W. Hensgen, A. Kamont, K. Kazaniecki, W. B. Johnson, P. W. Jones, G. Pisier, G. Racher, M. Rzeszut, G. Schechtman, Th. Schlumprecht, M. Wojciechowski, and P. Wojtaszczyk for most valuable feedback and discussions.

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