Cambridge University Press & Assessment
978-1-108-83867-2 — Hardy Martingales
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Excerpt
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Stochastic Holomorphy

Here, we cover basic results in stochastic holomorphy that form the foundation of our work in later chapters of this book. The main focus lies on the space of p-integrable holomorphic random variables, $H^p(\Omega)$, and on $SL^\infty(\Omega)$, consisting of complex Brownian martingales with uniformly bounded quadratic variation process.

Sections 1.2 and 1.3 contain estimates for the martingale maximal function, the quadratic variation process, and the Marcinkiewicz decomposition for holomorphic random variables in $H^1(\Omega)$. We investigate the stochastic Hilbert transform $\mathcal{H}$, the Doob projection N, and the martingale embedding operator M in considerable detail in Sections 1.4–1.6. The relation

$$\mathrm{Id} = NM$$

forms the basis for most applications of stochastic holomorphy to complex analysis. It is exploited in Section 1.7 and throughout the course of this book.

Finally, in Section 1.8 we prove that Doob's projection preserves square function estimates and that

$$N\colon SL^\infty(\Omega) \to SL^\infty(\mathbb{T})$$

forms a bounded linear surjection, inverting the action of the martingale embedding operator $M\colon SL^\infty(\mathbb{T}) \to SL^\infty(\Omega)$.

1.1 Preliminaries

Our preliminary section forms a listing of standard concepts in real, complex, and stochastic analysis, all of which are accessible by means of graduate-level textbooks. The following topics are reviewed in the subsections below:

(i) Conditional expectation, discrete time martingales, almost sure convergence, uniform integrability and convergence in L^1, Doob's inequalities, Davis decomposition.
(ii) Complex Brownian motion, Brownian martingales, stochastic integrals, Itô's formula in complex format.
(iii) Harmonic and subharmonic functions in the unit disk, the boundary convergence theorems of Fatou and Littlewood, the Poisson kernel and Kakutani's theorem, Green's function, and the occupation time formula for Brownian motion.
(iv) Convolution operators, the de La Valle Poussin kernels, the Fejer kernels, Hardy spaces in the unit disk, the Riesz factorisation, the F. and M. Riesz theorem, the Hilbert transform, the predual of $H^1(\mathbb{T})$, and polydisk algebras.
(v) Weak topology, reflexive Banach spaces, uniform integrability, weak compactness in L^1, and the Dunford–Pettis theorem.

1.1.1 Conditional Expectation and Martingales

In this section, we fix notation and recall basic concepts of measure and integration such as conditional expectation, filtered measure spaces, martingale sequences, uniform integrability, Doob's convergence theorems, and maximal inequalities. Basic references include the books by Kahane (1985) and Neveu (1975).

Measure distance: Let $(F, \mathcal{F}, \mu)$ be a finite measure space. We denote the linear space of $\mathcal{F}$-measurable scalar-valued functions by $\mathcal{L}^0 = \mathcal{L}^0(F, \mathcal{F}, \mu)$. We put

$$\mathcal{N} = \mathcal{N}(F, \mathcal{F}, \mu) = \left\{ f \in \mathcal{L}^0 : \mu(\{f \neq 0\}) = 0 \right\}.$$

Given $f, g \in \mathcal{L}^0$, we define the measure distance by

$$d_\mu(f, g) = \inf\{\epsilon > 0 : \mu(\{\omega \in F : |f(\omega) - g(\omega)| > \epsilon\}) < \epsilon\}.$$

We have $d_\mu(f, g) = 0$ if and only if $f - g \in \mathcal{N}$. The quotient space

$$L^0(F, \mathcal{F}, \mu) = \mathcal{L}^0 / \mathcal{N},$$

equipped with the distance induced by d_μ, becomes a complete metric space. When convenient (and feasible) we suppress the explicit dependence on the measure space $(F, \mathcal{F}, \mu)$ and compress our notation to $L^0 = L^0(F, \mathcal{F}, \mu)$.

Continuous operators on L^0: Suppose we are given a second measure space (S, Σ, λ), defining $L^0(S, \Sigma, \lambda)$ of equivalence classes of Σ-measurable functions $f: S \to \mathbb{R}$. Consider now a linear operator,

$$T: L^0(F, \mathcal{F}, \mu) \to L^0(S, \Sigma, \lambda).$$

The continuity of T, with respect to the metrics d_μ and d_λ, is equivalent to the following condition: For any $\epsilon > 0$ there exists $\delta > 0$ such that

$$\mu(\{\omega \in F: |f(\omega)| > \delta\}) < \delta \quad \text{implies} \quad \lambda(\{s \in S: |(Tf)(s)| > \epsilon\}) < \epsilon,$$

for any $f \in L^0(F, \mathcal{F}, \mu)$.

Lebesgue spaces: For $1 \leq p < \infty$, we let $\mathcal{L}^p(F, \mathcal{F}, \mu)$ denote the space of $\mathcal{F}$-measurable, p-integrable, scalar-valued functions equipped with the seminorm

$$\|f\|_p = \left(\int_F |f|^p d\mu \right)^{1/p}.$$

For $1 \leq p < \infty$, the quotient space

$$L^p(F, \mathcal{F}, \mu) = \mathcal{L}^p(F, \mathcal{F}, \mu)/\mathcal{N},$$

equipped with its canonical quotient norm, forms a Banach space called the Lebesgue space of p-integrable functions. Similarly, we form the Banach space

$$L^\infty(F, \mathcal{F}, \mu) = \mathcal{L}^\infty(F, \mathcal{F}, \mu)/\mathcal{N},$$

where $\mathcal{L}^\infty(F, \mathcal{F}, \mu)$ denotes the space of $\mathcal{F}$-measurable functions that are μ-essentially bounded. The space $\mathcal{L}^\infty(F, \mathcal{F}, \mu)$ is equipped with the seminorm, given by the essential supremum,

$$\|f\|_\infty = \mu - \operatorname{ess\,sup}_{\omega \in F} |f(\omega)| = \inf\{t > 0: \mu\{|f| > t\} = 0\}.$$

We frequently shorten the notation of $L^p(F, \mathcal{F}, \mu)$ to $L^p(F)$ and further to L^p, when the context allows us to do so.

Conditional Expectation

We fix a finite measure space $(F, \mathcal{F}, \mu)$ and a sub-sigma-algebra $\mathcal{G}$ of $\mathcal{F}$. For $f \in L^1(F, \mathcal{F}, \mu)$, we consider the set function

$$\lambda(A) = \int_A f d\mu, \quad A \in \mathcal{G}.$$

In view of Lebesgue's theorem on dominated convergence, $\lambda: \mathcal{G} \to \mathbb{R}$ is a finite signed measure on the measurable space $(F, \mathcal{G})$. Moreover, by the absolute continuity of the Lebesgue integral, for any sequence $A_n \in \mathcal{G}$ satisfying

$\lim_{n\to\infty} \mu(A_n) = 0$, we obtain $\lim_{n\to\infty} \lambda(A_n) = 0$. The Radon–Nikodym theorem asserts that there exists a $\mathcal{G}$-measurable function $g\colon F \to \mathbb{R}$ such that

$$\lambda(A) = \int_A g d\mu, \quad A \in \mathcal{G}, \quad \text{and} \quad \int_F |g| d\mu \le \int_F |f| d\mu.$$

Moreover, up to $\mathcal{G}$-measurable sets of vanishing μ-measure, g is uniquely determined. We say that g is the conditional expectation of f with respect to $\mathcal{G}$ and write

$$\mathbb{E}(f|\mathcal{G}) = g.$$

Clearly, taking the conditional expectation defines a linear operation, and $f \ge 0$ implies $\mathbb{E}(f|\mathcal{G}) \ge 0$. Moreover, $\mathbb{E}(1_F|\mathcal{G}) = 1_F$ and

$$\|\mathbb{E}(f|\mathcal{G})\|_p \le \|f\|_p,$$

for $f \in L^p(F)$, $1 \le p \le \infty$. We frequently use the following properties:

(i) If $f, h \in L^1(F)$ and h is $\mathcal{G}$-measurable, then $\mathbb{E}(hf|\mathcal{G}) = h\mathbb{E}(f|\mathcal{G})$.
(ii) If $\mathcal{G}_1$ is a sub-sigma-algebra of $\mathcal{G}$, then $\mathbb{E}(\mathbb{E}(f|\mathcal{G})|\mathcal{G}_1) = \mathbb{E}(f|\mathcal{G}_1)$ for $f \in L^1(F)$.

Let $(\mathcal{G}_k)_{k=0}^{\infty}$ be a sequence of sub-sigma-algebras of $\mathcal{F}$ satisfying $\mathcal{G}_{k-1} \subseteq \mathcal{G}_k \subseteq \mathcal{F}$ for $k \in \mathbb{N}$, and $\mathcal{G}_0 = \{\emptyset, F\}$. Let $\mathcal{G}$ denote the sigma-algebra generated by $\bigcup \mathcal{G}_k$, and let $\mu_{\mathcal{G}}$ be the restriction of μ to $\mathcal{G}$. We then say that $(F, (\mathcal{G}_k), \mu_{\mathcal{G}})$ forms a filtered finite measure space.

The following far-reaching theorem addresses convergence of the conditional expectations $\mathbb{E}(f|\mathcal{G}_k)$. As it turns out, for $f \in L^1$, we have convergence in the L^1-norm and point-wise convergence almost everywhere.

Theorem 1.1.1 *For $f \in L^1(F)$, set $g = \mathbb{E}(f|\mathcal{G})$ and $f_k = \mathbb{E}(f|\mathcal{G}_k)$. Then*

$$\lim_{k\to\infty} \int_F |g - f_k| d\mu = 0,$$

and there exists $E \subset F$ satisfying $\mu(E) = 0$ such that $g(\omega) = \lim_{k\to\infty} f_k(\omega)$, for $\omega \in F \setminus E$.

Examples

The canonical product filtration on the infinite torus product forms a concrete realization of a filtered probability space. We explicitly describe the dyadic filtration embedded in the infinite torus product.

Example 1.1.1 (The product filtration on $\mathbb{T}^{\mathbb{N}}$) Let $\mathbb{T} = \{e^{i\theta} : \theta \in [0, 2\pi[\}$ be equipped with the Lebesgue sigma-algebra and normalized angular measure, denoted dm. Thus m is the Haar measure on $\mathbb{T}$, i.e., the unique rotation invariant probability measure on the sigma-algebra of Lebesgue measurable subsets of $\mathbb{T}$. We denote by

$$\mathbb{T}^{\mathbb{N}} = \left\{(z_i)_{i=1}^{\infty} : z_i \in \mathbb{T}\right\}$$

its countable product equipped with the product sigma-algebra and normalized product Haar measure, denoted $\mathbb{P}$. We denote by $\mathcal{F}_k$ the sigma-algebra on $\mathbb{T}^{\mathbb{N}}$ generated by the cylinder sets

$$\left\{\left(A_1, \dots, A_k, \mathbb{T}^{\mathbb{N}}\right)\right\},$$

where A_i, $i \le k$, are measurable subsets of $\mathbb{T}$. A measurable function F defined on $\mathbb{T}^{\mathbb{N}}$ is measurable with respect to $\mathcal{F}_k$ if it depends only on the first k variables of $\mathbb{T}^{\mathbb{N}}$. The conditional expectation with respect to the sigma-algebra $\mathcal{F}_k$ acts as integration with respect to the variable z_i, where $i \ge k+1$. Explicitly, if $f \in L^1(\mathbb{T}^{\mathbb{N}})$, then for almost every $x \in \mathbb{T}^k$ we have

$$\mathbb{E}(f|\mathcal{F}_k)(x) = \int_{\mathbb{T}^{\mathbb{N}}} f(x, z) d\mathbb{P}(z). \tag{1.1.1}$$

The filtered probability space $(\mathbb{T}^{\mathbb{N}}, (\mathcal{F}_k), \mathbb{P})$ is our preferred framework for discussing discrete-time martingales.

Example 1.1.2 (The dyadic filtration on $\mathbb{T}^{\mathbb{N}}$) We define the independent Rademacher functions $\sigma_k : \mathbb{T}^{\mathbb{N}} \to \{-1, 1\}$, by

$$\sigma_k(z) = \operatorname{sign}(\cos_k(z)),$$

where $z = (z_k)$ and $\cos_k(z) = \Re z_k$. Set $\mathcal{D}_k$ to be the finite algebra of subsets in $\mathbb{T}^{\mathbb{N}}$ generated by $\sigma_1, \dots, \sigma_k$. Let $\mathcal{D}$ be the sigma-algebra generated by $\bigcup \mathcal{D}_k$. Denoting $\mathbb{P}_{\mathcal{D}}$ to be the restriction of $\mathbb{P}$ to $\mathcal{D}$, we obtain the filtered probability space $(\mathbb{T}^{\mathbb{N}}, (\mathcal{D}_k), \mathbb{P}_{\mathcal{D}})$.

For $1 \le p \le \infty$, the space $L^p(\mathbb{T}^{\mathbb{N}}, \mathcal{D}, \mathbb{P}_{\mathcal{D}})$ is closed in $L^p(\mathbb{T}^{\mathbb{N}}, \mathcal{F}, \mathbb{P})$. The conditional expectation operator

$$\mathbb{E}(\cdot \,|\, \mathcal{D}) : L^p(\mathbb{T}^{\mathbb{N}}, \mathcal{F}, \mathbb{P}) \to L^p(\mathbb{T}^{\mathbb{N}}, \mathcal{D}, \mathbb{P}_{\mathcal{D}})$$

is a surjective, idempotent contraction for $1 \le p \le \infty$.

Example 1.1.3 (The dyadic filtration in $[0, 1[$) We say that $I \subseteq [0, 1[$ is a dyadic interval if there exists $n \in \mathbb{N} \cup \{0\}$ and $1 \le i \le 2^n$ such that $I = [(i-1)/2^n, i/2^n[$. We let $\mathcal{I}$ denote the collection of all dyadic intervals contained in the unit interval $[0,1[$. For $I \in \mathcal{I}$ there exist uniquely determined $I_1, I_2 \in \mathcal{I}$ such that $I_1 \cap I_2 = \emptyset$ and $I_1 \cup I_2 = I$. In that case $|I_1| = |I_2| = |I|/2$. If $\inf I_1 = \inf I$ we say that I_1 is the left half of I, and

$$h_I = 1_{I_1} - 1_{I_2}, \tag{1.1.2}$$

is called the L^∞-normalized Haar function, supported on I. Let $\mathcal{I}_k = \{I \in \mathcal{I} : |I| = 2^{-k}\}$ and let $\mathcal{G}_k = \sigma\{\mathcal{I}_k\}$ denote the sigma-algebra generated by $\mathcal{I}_k$. For $f \in L^1[0,1[$, we have

$$\mathbb{E}(f|\mathcal{G}_k) = \sum_{I \in \mathcal{I}_k} \int_I f \frac{dt}{|I|} 1_I, \tag{1.1.3}$$

and

$$\mathbb{E}(f|\mathcal{G}_{k+1}) - \mathbb{E}(f|\mathcal{G}_k) = \sum_{I \in \mathcal{I}_k} \int_I f h_I \frac{dt}{|I|} h_I. \tag{1.1.4}$$

With $([0,1[, \mathcal{L}, \lambda)$ denoting Lebesgue's measure space, a filtered probability space is formed by $([0,1[, (\mathcal{G}_k), \lambda)$.

Dyadic intervals $\mathcal{I}$ are canonically ordered as follows. If $I, J \in \mathcal{I}$ and $|I| \le |J|$ then $I \prec J$; if $|I| = |J|$ and $\inf I < \inf J$ then $I \prec J$. The canonical order on $\mathcal{I}$ is often called the lexicographic order for obvious reasons.

Martingale Convergence

We fix again a finite measure space $(F, \mathcal{F}, \mu)$, and let $(\mathcal{G}_k)$ be a sequence of increasing sub-sigma-algebras of $\mathcal{F}$, satisfying $\mathcal{G}_0 = \{\emptyset, F\}$. We say that a sequence of μ-integrable functions (f_k) is a $(\mathcal{G}_k)$-martingale if

$$f_k = \mathbb{E}(f_{k+1}|\mathcal{G}_k), \quad k \in \mathbb{N}. \tag{1.1.5}$$

We write $\Delta f_k = f_k - f_{k-1}$ and say that (Δf_k) is the martingale difference sequence of (f_k). Should there exist $f \in L^1(F)$ such that $\mathbb{E}(f|\mathcal{G}_k) = f_k$, we say that the martingale (f_k) is closed by f.

The following theorems, due to Doob, assert that the martingale condition (1.1.5) imposes considerable constraints on the size and oscillations of an L^1-bounded sequence (f_k). In view of Theorem 1.1.1, a closed martingale converges in the norm of $L^1(F)$, and hence forms a uniformly integrable subset of $L^1(F)$. We recall that $\{f_k : k \in \mathbb{N}\}$ is (defined to be) uniformly integrable in $L^1(F)$, if

$$\lim_{t\to\infty} \sup_{k\in\mathbb{N}} \int_{\{|f_k|>t\}} |f_k| d\mu = 0.$$

The following theorem provides the converse implication, and thus sheds light on the relation between general martingales and closed ones.

Theorem 1.1.2 (Doob's martingale convergence theorem) *For any $(\mathcal{G}_k)$-martingale sequence (f_k), the following conditions are equivalent:*

(i) *There exists* $f \in L^1(F)$ *such that* $f_k = \mathbb{E}(f|\mathcal{G}_k)$ *for* $k \in \mathbb{N}$.
(ii) *The set* $\{f_k : k \in \mathbb{N}\}$ *is uniformly integrable in* $L^1(F)$.
(iii) *There exists* $f \in L^1(F)$ *such that* $\lim_{k\to\infty} \|f - f_k\|_1$.

We next address the question of almost sure convergence in the context of general submartingales. A sequence of μ-integrable functions (f_k) is a $(\mathcal{G}_k)$ submartingale if f_k is $\mathcal{G}_k$-measurable and

$$f_k \le \mathbb{E}(f_{k+1}|\mathcal{G}_k), \quad k \in \mathbb{N}.$$

Theorem 1.1.3 *For any* $L^1(F)$*-bounded submartingale* (f_k)*, there exists* $E \subset F$ *with* $\mu(E) = 0$ *such that*

$$f(\omega) = \lim_{k\to\infty} f_k(\omega),$$

exists for $\omega \in F\backslash E$. *If* $\{f_k\}$ *is a uniformly integrable subset of* $L^1(F)$*, then*

$$\lim_{k\to\infty} \|f - f_k\|_1.$$

Summarizing Theorems 1.1.2 and 1.1.3, every L^1-bounded martingale converges almost surely; however its convergence in the L^1-norm requires uniform integrability. By contrast, for $p > 1$, every L^p-bounded martingale is norm-convergent in L^p. This assertion is a direct consequence of Doob's maximal inequalities stated in Theorem 1.1.4.

Theorem 1.1.4 *Let* (f_k) *be an* $L^1(F)$*-bounded* $(\mathcal{G}_k)$*-martingale. Then*

$$t\mu\left\{\max_{k\le n} |f_k| > t\right\} \le \int_F |f_n| d\mu, \tag{1.1.6}$$

where $t > 0$, $n \in \mathbb{N}$. *If* $p > 1$ *and* (f_k) *is an* $L^p(F)$*-bounded* $(\mathcal{G}_k)$*-martingale then*

$$\int_F \max_{k\le n} |f_k|^p d\mu \le \left(\frac{p}{p-1}\right)^p \int_F |f_n|^p d\mu, \tag{1.1.7}$$

for $n \in \mathbb{N}$. *Inequalities* (1.1.6) *and* (1.1.7) *hold true if* (f_n) *is a nonnegative submartingale.*

In Burkholder's classical L^p-inequality, maximal functions are replaced by martingale square functions.

Theorem 1.1.5 (Burkholder's theorem) *For* $1 < p < \infty$*, there exist* $c_p > 0$ *and* $C_p < \infty$ *such that for any* $L^p(F)$*-bounded* $(\mathcal{G}_k)$*-martingale* (f_k) *then*

$$c_p^p \int_F |f_n|^p d\mu \le \int_F \left(|f_0|^2 + \sum_{k=1}^n |f_k - f_{k-1}|^2\right)^{p/2} d\mu \le C_p^p \int_F |f_n|^p d\mu, \tag{1.1.8}$$

where $f_0 = \mathbb{E}(f_n|\mathcal{G}_0)$ *and* $n \in \mathbb{N}$.

As $p \to 1$, the constants in Theorem 1.1.5 satisfy $c_p \to 0$ and $C_p \to \infty$. The limiting case, $p = 1$, in Doob's maximal inequality (1.1.7) and Burkholder's square function inequality (1.1.8) is captured by the following result from Davis (1970).

Theorem 1.1.6 (Davis's theorem) *For any $L^1(F)$-bounded $(\mathcal{G}_k)$-martingale (f_k),*

$$c_1 \int_F \max_{k \le n} |f_k| d\mu \le \int_F S(f_n) d\mu \le C_1 \int_F \max_{k \le n} |f_k| d\mu,$$

where $f_0 = \mathbb{E}(f_n|\mathcal{G}_0)$, $n \in \mathbb{N}$, and

$$S(f_n) = \left(|f_0|^2 + \sum_{k=1}^{n} |f_k - f_{k-1}|^2 \right)^{1/2}, \tag{1.1.9}$$

the constants $c_1 > 0$ and $C_1 < \infty$ are independent of the martingale and the underlying filtration.

The basic tool invented by Davis (1970) in the proof of Theorem 1.1.6 is referred to as the Davis decomposition of martingales.

Theorem 1.1.7 *For any $(\mathcal{G}_k)$-martingale (f_k), there exist $(\mathcal{G}_k)$-martingale sequences (g_k) and (b_k) satisfying*

$$f_n = g_n + b_n, \quad n \in \mathbb{N}, \tag{1.1.10}$$

$$|\Delta g_n| \le 8 \max_{k \le n-1} |f_k|, \quad n \in \mathbb{N}, \tag{1.1.11}$$

and

$$\sum_{k=1}^{n} \|\Delta b_k\|_{L^1(\mu)} \le 8 \|\max_{k \le n} |f_k|\|_{L^1(\mu)}, \quad n \in \mathbb{N}, \tag{1.1.12}$$

where $\Delta b_k = b_k - b_{k-1}$ and $\Delta g_k = g_k - g_{k-1}$.

For the specific constants in Theorem 1.1.7, we refer to Garsia (1973, Theorem III 3.5).

Martingale Spaces

Given a filtered measure space $(F, (\mathcal{G}_k), \mu)$, we define the martingale Hardy space $H^1(F, (\mathcal{G}_k), \mu)$ to consist of those $(\mathcal{G}_k)$-martingales $f = (f_k)$ for which

$$\left\|(f_k)\right\|_{H^1(F,(\mathcal{G}_k),\mu)} = \left\|S(f)\right\|_{L^1(F,\mu)} < \infty,$$

where

$$S(f) = \left(|f_0|^2 + \sum_{k=1}^{\infty} |f_k - f_{k-1}|^2 \right)^{1/2}.$$

The conditional square function of a $(\mathcal{G}_k)$-martingale $f = (f_k)$ is defined by

$$S_{\mathrm{cd}}(f) = \Big(|f_0|^2 + \sum \mathbb{E}_{k-1}|f_k - f_{k-1}|^2\Big)^{1/2}.$$

It gives rise to the space of predictable martingales $\mathcal{P}(F, (\mathcal{G}_k), \mu)$, which consists of those $(\mathcal{G}_k)$-martingales $f = (f_k)$ for which

$$\big\|(f_k)\big\|_{\mathcal{P}(F,(\mathcal{G}_k),\mu)} = \big\|S_{\mathrm{cd}}(f)\big\|_{L^1(F,\mu)} < \infty. \tag{1.1.13}$$

In Section 2.5 we will prove the Burkholder–Gundy inequality, which asserts that

$$\big\|(f_k)\big\|_{H^1(F,(\mathcal{G}_k),\mu)} \le 2\big\|(f_k)\big\|_{\mathcal{P}(F,(\mathcal{G}_k),\mu)}. \tag{1.1.14}$$

Martingales in the Hardy space $H^1(F, (\mathcal{G}_k), \mu)$ and in $\mathcal{P}(F, (\mathcal{G}_k), \mu)$ converge almost surely and in L^1; they may thus be identified with their almost sure limits.

Theorem 1.1.8 *Each martingale $(f_k) \in H^1(F, (\mathcal{G}_k), \mu)$ is closed. That is, there exists $f \in L^1(F, \mu)$, such that*

$$\lim_{k\to\infty} \big\|f_k - f\big\|_{L^1} = 0,$$

and $f_k = \mathbb{E}(f|\mathcal{G}_k)$, for $k \in \mathbb{N}$. The same conclusion holds true for martingales $(f_k) \in \mathcal{P}(F, (\mathcal{G}_k), \mu)$.

Proof By Davis's theorem (Theorem 1.1.6), each martingale $(f_k) \in H^1(F, (\mathcal{G}_k), \mu)$ is a uniformly integrable subset of $L^1(F, \mu)$. Hence Doob's martingale convergence theorem (Theorem 1.1.2) shows that the sequence (f_k) converges in $L^1(F, \mu)$, and that there exists $f \in L^1(F, \mu)$, measurable with respect to $\mathcal{G} = \sigma(\bigcup \mathcal{G}_k)$, such that $f_k = \mathbb{E}(f|\mathcal{G}_k)$, for $k \in \mathbb{N}$.

By Inequality (1.1.14) any martingale in $\mathcal{P}(F, (\mathcal{G}_k), \mu)$ is contained in $H^1(F, (\mathcal{G}_k), \mu)$. Hence, the conclusion of the theorem holds for $\mathcal{P}(F, (\mathcal{G}_k), \mu)$. □

Operators of Weak Type (1:1)

The space $L^{1,\infty} = L^{1,\infty}(F, \mathcal{F}, \mu)$ consists of those $\mathcal{F}$-measurable $f : F \to \mathbb{C}$ for which

$$\|f\|_{1,\infty} = \sup_{t>0} t\mu\{|f| > t\} < \infty.$$

We say that a sublinear operator $T : L^1(F, \mathcal{F}, \mu) \to L^{1,\infty}(F, \mathcal{F}, \mu)$ is of weak type (1:1) if there exists $C < \infty$ such that

$$\|T(f)\|_{1,\infty} \le C\|f\|_1, \quad f \in L^1. \tag{1.1.15}$$

We repeatedly use that a sublinear operator T is of weak type (1:1) if and only if for any $f \in L^1(F, \mathcal{F}, \mu)$ and $\varphi \in L^\infty(F, \mathcal{F}, \mu)$ with $\|\varphi\|_\infty \le 1$, we have

$$\int_F |T(f)|^{1/2} |\varphi| d\mu \le C_1 \left(\int_F |f| d\mu \cdot \int_F |\varphi| d\mu \right)^{1/2}, \tag{1.1.16}$$

for some $C_1 < \infty$. We refer to Wojtaszczyk's book (1991, Lemma III.I.11) for the equivalence between Inequalities (1.1.15) and (1.1.16).

1.1.2 Brownian Martingales

We review basic results on Brownian martingales, stopping times, martingale convergence theorems, stochastic integrals, complex Brownian motion, and Itô's formula and some of its numerous consequences. These concepts are covered, for example, in the books by Revuz and Yor (1991), Bass (1995), and Durrett (1984).

Brownian Motion

Let $(\Omega, \mathcal{F}, \mathbb{P})$ be a fixed probability space. Let $\mathbb{R}_0^+ = \{t \in \mathbb{R} : t \ge 0\}$. A map $X\colon \Omega \times \mathbb{R}_0^+ \to \mathbb{R}$ defines uniquely maps

$$X_t\colon \Omega \to \mathbb{R}, \qquad \omega \to X(\omega, t),$$

for $t \ge 0$. We say that $X = (X_t : 0 \le t < \infty)$ forms a real-valued stochastic process if each of the maps X_t is $\mathcal{F} - \mathcal{B}$ measurable where $\mathcal{B}$ denotes the Borel sigma-algebra on $\mathbb{R}$. Similarly we define complex-valued processes, and processes indexed by $\mathbb{N}_0$.

A real stochastic process $(x_t : 0 \le t < \infty)$ on a probability space $(\Omega, \mathcal{F}, \mathbb{P})$ is called one-dimensional Brownian motion (or simply Brownian motion) if

(i) $x_0(\omega) = 0$ for almost every $\omega \in \Omega$.
(ii) For $t \in \mathbb{R}_0^+$ and $h > 0$, the increment $x_{t+h} - x_t$ is independent of the process up to time t, $(x_s : 0 < s \le t)$, and Gaussian distributed with mean 0 and variance h; that is, for every measurable $A \subseteq \mathbb{R}$,

$$\mathbb{P}(\{x_{t+h} - x_t \in A\}) = \int_A e^{-x^2/2h} \frac{dx}{\sqrt{2\pi h}}.$$

(iii) The function $t \to x_t(\omega)$ is continuous, for almost every $\omega \in \Omega$.

We let $\mathcal{F}_t$ denote the completion, with respect to $(\Omega, \mathcal{F}, \mathbb{P})$, of the sigma-algebra generated by $(x_s : 0 < s \le t)$. We have (see for instance Bass [1995, p. 17–18]) that

$$\mathcal{F}_t = \bigcap_{\epsilon > 0} \mathcal{F}_{t+\epsilon}, \tag{1.1.17}$$