

Contents

Preface	xiii
Notation and Acronyms	xvii
I FOUNDATIONS	1
1 Introduction	3
1.1 A Very Brief History of Time	3
1.2 Dynamical Cores	4
1.3 Gravity	5
1.4 Fluid Composition of Earth's Atmosphere and Oceans	6
1.5 Organisation of Chapters	12
2 Governing Equations for Motion of a Dry Atmosphere: Vector Form	14
2.1 Preamble	14
2.2 Fluid Dynamics	15
2.3 Thermodynamics of an Ideal Gas	29
2.4 The Governing Equations for Motion of an Ideal Gas	36
2.5 Concluding Remarks	37
3 Governing Equations for Motion of a Cloudy Atmosphere: Vector Form	38
3.1 Preamble	38
3.2 Representation of Water and Other Substances in the Atmosphere	39
3.3 The Equation of State for Cloudy Air	49
3.4 The Momentum Equation for Cloudy Air	52
3.5 The Thermodynamic-Energy Equation for Cloudy Air	53
3.6 Alternative Forms for the Thermodynamic-Energy Equation	56
3.7 The Governing Equations for Motion of a Cloudy Atmosphere	62
3.8 Concluding Remarks	63
Appendix: Derivation of the Equation of State for Cloudy Air from First Principles	65

4	Governing Equations for Motion of Geophysical Fluids: Vector Form	68
4.1	Preamble	68
4.2	More General Thermodynamics	69
4.3	Functional Forms for a Cloudy-Air Parcel	86
4.4	The Governing Equations for Motion of a Geophysical Fluid	92
4.5	Concluding Remarks	93
	Appendix: Specific Heat Capacities for an Ideal Gas	94
5	Orthogonal-Curvilinear Coordinate Systems	97
5.1	Preamble	97
5.2	Deep Orthogonal-Curvilinear Coordinates	98
5.3	Shallow Orthogonal-Curvilinear Coordinates	111
5.4	Concluding Remarks	118
6	Governing Equations for Motion of Geophysical Fluids: Curvilinear Form	119
6.1	Preamble	119
6.2	The Governing Equations in Vector Form	120
6.3	Axial-Orthogonal-Curvilinear Coordinates	121
6.4	Coriolis Terms	122
6.5	The Governing Equations in Axial-Orthogonal-Curvilinear Coordinates	126
6.6	The Governing Equations in Spherical-Polar and Cylindrical-Polar Coordinates	128
6.7	Euler–Lagrange Forms of the Momentum Components	128
6.8	Concluding Remarks	129
	Appendix A: The Governing Equations in Spherical-Polar Coordinates	130
	Appendix B: The Governing Equations in Cylindrical-Polar Coordinates	131
7	Representation of Gravity: Basic Theory and Spherical Planets	133
7.1	Preamble	133
7.2	A Guide to This Chapter and to the Next One	141
7.3	Equilibrium States for Unaccelerated Flow	143
7.4	The Geopotential at and near Earth’s Surface	147
7.5	Newtonian Gravity and Potential Theory	149
7.6	A Spherical Planet of Constant Density	150
7.7	Avenues for Investigation	156
7.8	A Spherical Planet of Variable Density	156
7.9	Concluding Remarks	171
	Appendix: Some Spherical Relations	172
8	Representation of Gravity: Further Theory and Spheroidal Planets	174
8.1	Preamble	174
8.2	Functional Forms for Spheroidal Planets	176
8.3	An Ellipsoidal Planet of Constant Density	182
8.4	Reformulation of the Procedure to Determine Newtonian Gravity Outside a Planet	189
8.5	An Ellipsoidal Planet of Variable Density	191
8.6	Spherical Geopotential Approximation as an Asymptotic Limit	211
8.7	Concluding Remarks	214

9	Thermodynamic Potentials and Thermodynamical Consistency	216
9.1	Preamble	216
9.2	Thermodynamic Potentials	218
9.3	Basic Gibbs Thermodynamic Potentials	223
9.4	Composite Gibbs Potentials	228
9.5	Concluding Remarks	237
10	Moist Thermodynamics	238
10.1	Preamble	238
10.2	Humid Air	239
10.3	Latent Internal Energy and Phase Transitions	243
10.4	Water Substance in a Vacuum	247
10.5	Cloudy Air, Possibly Containing Liquid and/or Frozen Water	259
10.6	The Triple Point of Water in the Presence of Dry Air	275
10.7	Definition of Some Thermodynamic Quantities	286
10.8	Concluding Remarks	292
11	Ocean Thermodynamics	294
11.1	Preamble	294
11.2	Oceanic Gibbs Potentials	296
11.3	Derivation of Vallis (2017)'s Prototypical Gibbs Potential	299
11.4	An Alternative Prototypical Gibbs Potential for an Ocean	312
11.5	The TEOS-10 Gibbs Potential	319
11.6	Concluding Remarks	327
12	Geopotential Coordinates for Modelling Planetary Atmospheres and Oceans	328
12.1	Preamble	328
12.2	Geodesy and the World Geodetic System	330
12.3	The Classical Spherical Geopotential Approximation Revisited	338
12.4	Geopotential Approximation for Ellipsoidal Planets	340
12.5	Further Geopotential Approximation above Earth's Geoid	345
12.6	Further Geopotential Approximation below Earth's Geoid	349
12.7	Interlude	353
12.8	Orthogonal Trajectories to the Geopotential Surfaces	354
12.9	GREAT Coordinates	357
12.10	Concluding Remarks	366
	Appendix: The Equilibrium Depth of an Ocean Covering a Planet	369
13	Vertical Coordinates and Boundary Conditions	371
13.1	Preamble	371
13.2	The Deep-Fluid Equations and Boundary Conditions	373
13.3	Mass Conservation	388
13.4	Energetics	389
13.5	Axial-Angular-Momentum Conservation	393
13.6	Boundary Conditions in the Vertical and Global Conservation	395
13.7	Conservation with the Shallow Approximation	403
13.8	An Energy-Like Invariant for Elastic Lids at Finite Pressure	403
13.9	An Atmospheric State with Zero Pressure at Finite Height	405
13.10	Concluding Remarks	410
	Appendix: Some Useful Identities	411

14	Variational Methods and Hamilton's Principle of Stationary Action	413
14.1	Preamble	413
14.2	Eulerian versus Lagrangian Viewpoints for Fluid Dynamics	414
14.3	Mass Conservation	417
14.4	Functionals and Variational Principles	419
14.5	Hamilton's Principle of Stationary Action	427
14.6	Gravitational Attraction between Two Particles Revisited	431
14.7	A System of Point Particles	435
14.8	Governing Equations for Global Fluids: Vector Form	439
14.9	Governing Equations for Global Fluids: Curvilinear Form	449
14.10	Euler–Lagrange Equations for Global Fluids	452
14.11	Concluding Remarks	456
	Appendix: Variations in Axial-Orthogonal-Curvilinear Coordinates	456
15	Conservation	463
15.1	Preamble	463
15.2	Governing Equations	464
15.3	Conservation Principles: Vector Form	465
15.4	Conservation Principles: Curvilinear Form	480
15.5	Noether's Theorem, Symmetries, and Conservation	487
15.6	Concluding Remarks	494
II	DYNAMICALLY CONSISTENT EQUATION SETS	495
16	Deep and Shallow, Dynamically Consistent Equation Sets in 3D	497
16.1	Preamble	497
16.2	A Unified Quartet of Dynamically Consistent Equation Sets	500
16.3	Derivation Methodologies for Approximate Equation Sets	511
16.4	Classical Eulerian Derivation	511
16.5	Lagrangian Derivation Using Hamilton's Principle	512
16.6	Lagrangian Derivation Using Euler–Lagrange Equations	516
16.7	Equation Transition from Deep Fluids to Shallow Fluids	517
16.8	Concluding Remarks	520
	Appendix A: Four Equation Sets in Spherical-Polar Coordinates	522
	Appendix B: Four Equation Sets in Axial-Orthogonal-Curvilinear Coordinates	526
17	Quasi-Shallow, Dynamically Consistent Equation Sets in 3D	531
17.1	Preamble	531
17.2	Classical Eulerian Derivation	533
17.3	Lagrangian Derivation	543
17.4	A Unified Sextet of Equation Sets in Spheroidal Geometry	549
17.5	A Unified Sextet of Equation Sets in Spherical Geometry	553
17.6	Concluding Remarks	557
	Appendix A: Quasi-Shallow Equation Sets in Axial-Orthogonal-Curvilinear Coordinates	557
	Appendix B: Variations for Quasi-Shallow Contributions	559
18	Shallow-Water Equation Sets in 2D	562
18.1	Preamble	562
18.2	Eulerian Derivation of the Basic Shallow-Water Equations	566

18.3	Horizontal Coordinate Systems and Models of Gravity	575
18.4	Lagrangian Density for the Basic Shallow-Water Equations	584
18.5	Quasi-Shallow Enhancement of Lagrangian Density	591
18.6	Euler–Lagrange Derivation of the Quasi-Shallow Enhanced Set	595
18.7	‘Quasi-Shallow’ Shallow-Water Conservation Principles	599
18.8	The ‘Quasi-Shallow’ Shallow-Water Equations in Spherical Geometry	602
18.9	Derivation of a Unified Quartet of Equation Sets	605
18.10	The Unified Quartet in Spherical Geometry	614
18.11	Concluding Remarks	618
	Appendix A: Derivation of 2D Quasi-Shallow Lagrangian Density by Vertically Averaging the 3D One	621
	Appendix B: Conservation Principles for the ‘Quasi-Shallow’ Shallow-Water Equations	624
19	A Barotropic Potential Vorticity (BPV) Equation for Flow over a Spheroidal Planet	631
19.1	Preamble	631
19.2	The Momentum and Mass-Continuity Equations in Curvilinear Form	632
19.3	Inviscid, Horizontal, Shallow Flow in Spheroidal Geometry	634
19.4	Global Conservation	639
19.5	The BPV Equation for a Spheroidal Planet	643
19.6	An Alternative Derivation of the BPV Equation	646
19.7	Dynamical Consistency	647
19.8	The Poisson Problem for Pressure	648
19.9	Variational Derivation of the Momentum Equations	649
19.10	Concluding Remarks	650
III	EXACT STEADY AND UNSTEADY NON-LINEAR SOLUTIONS	653
20	Exact Steady Solutions of the Global Shallow-Water Equations	655
20.1	Preamble	655
20.2	The Shallow-Water Equations in Spheroidal Geometry	657
20.3	A Derivation Methodology	658
20.4	A Physical Interpretation of $h_1^S(\xi_2)$	659
20.5	Some Illustrative Solutions	661
20.6	Rotated Solutions in Spherical Geometry	668
20.7	Interlude	669
20.8	The Stability of Exact Solutions to Linear Perturbation	671
20.9	Illustrative Examples of the Application of the Stability Analysis	682
20.10	Concluding Remarks	695
	Appendix: Rotated Coordinate Transformations	695
21	Exact 3D Steady Solutions of Global Equation Sets	698
21.1	Preamble	698
21.2	A Unified Quartet of Governing Equations	699
21.3	Simplification for Steady, Axially Symmetric Flow	700
21.4	Compatibility Constraints for Balance	701
21.5	A Change of Dependent Variable	703
21.6	Construction of Exact Steady Solutions	705
21.7	A Generalised Thermal-Wind Equation	708
21.8	Three Illustrative Examples	710
21.9	Concluding Remarks	721

22	Exact Unsteady Solutions of the Barotropic Potential Vorticity Equation over an Ellipsoid	723
22.1	Preamble	723
22.2	Derivation of Exact Unsteady Solutions	725
22.3	A Complementary Derivation of Exact Unsteady Solutions	737
22.4	Diagnosis of Pressure for a Particular Solution	739
22.5	Diagnosis of Pressure for a Family of Solutions	742
22.6	Concluding Remarks	744
23	Exact Unsteady Solutions in 3D over an Ellipsoidal Planet	746
23.1	Preamble	746
23.2	A Quartet of Equation Sets for Unforced 3D Fluid Flow over a Rotating Ellipsoidal Planet	747
23.3	Preparatory Steps	750
23.4	Exact Barotropic Solutions over an Ellipsoid	756
23.5	A Family of Exact, Unsteady 3D Solutions	760
23.6	A Particular Exact, Unsteady, 3D Solution	761
23.7	The Top Boundary Condition	762
23.8	Test Cases for Validating 3D Dynamical Cores	762
23.9	Concluding Remarks	771
	Appendix: Vector Identities	773
	References	777
	Index	784