Global Atmospheric and Oceanic Modelling

Combining rigorous theory with practical application, this book provides a unified and detailed account of the fundamental equations governing atmospheric and oceanic fluid flow on which global, quantitative models of weather and climate prediction are founded. It lays the foundation for more accurate models by making fewer approximations and imposing dynamical and thermodynamical consistency, moving beyond the assumption that the Earth is perfectly spherical. A general set of equations is developed in a standard notation with clearly stated assumptions, limitations, and important properties. Some exact, non-linear solutions are developed to promote further understanding and for testing purposes. This book contains a thorough consideration of the fundamental equations for atmospheric and oceanic models, and is therefore invaluable to both theoreticians and numerical modellers. It also stands as an accessible source for reference purposes.

Andrew N. Staniforth – now retired – led the development of dynamical cores for weather and climate prediction at two national centres (Canada and the UK). He has published over 100 peer-reviewed journal articles and is the recipient of various prizes and awards including the Editor's Award (American Meteorological Society, 1990); the Andrew Thompson Prize (Canadian Meteorological and Oceanographic Society, 1993); and the Buchan and Adrian Gill Prizes (Royal Meteorological Society, 2007 and 2009).

‘Well, this is an impressive book. It covers both the equations of motion and how those equations and their approximations can be used in models of the ocean and atmosphere. It is clearly written, careful and thorough, with a range and a depth that is unmatched elsewhere. It will be of immense value both to those interested in the fundamentals and to those wishing to build models that have a sound foundation. It will be a standard for years to come.’

Geoffrey K. Vallis, University of Exeter

‘This is the textbook I wish I’d had as a graduate student and course instructor! This is an incredibly comprehensive resource for students and researchers alike. I am confident the book will become the go-to reference on atmospheric and oceanic modelling for the 2020s and beyond.’

Andrew Weaver, University of Victoria

‘Andrew Staniforth has produced a comprehensive and insightful book on the mathematical foundation of global atmosphere and oceanic modelling. For different geophysical fluid applications, he guides us masterfully from the first principles of fluid physics to their evolution equations. The book covers all the fundamental aspects of these equations including conservation laws and exact non-linear solutions. This brilliant book is ideal for introducing graduate students to the subject matter as much as it is relevant for experts as a reference book.’

Gilbert Brunet, Bureau of Meteorology, Melbourne

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Global Atmospheric and Oceanic Modelling

Fundamental Equations

Andrew N. Staniforth

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To the memory of my parents,
who so strongly encouraged me to pursue the educational
opportunities available to me, but not to them.

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Preface

Purpose For novices and experts alike, this book aims to provide a unified, consolidated and detailed account of the underlying geophysical-fluid-dynamical equations on which global, quantitative, atmospheric, and oceanic models for climate and weather prediction are founded. Everything is built upon, or around, these governing equations. To accomplish this goal:

- A *general* set of governing equations is developed in a standard notation, with clearly stated assumptions, limitations, and important properties (in Part I).
- *Approximated* equation sets are related to this general set, again with clearly stated assumptions, limitations, and important properties (in Part II).
- Some *exact*, steady and unsteady, non-linear solutions are developed for testing purposes and to promote further understanding (in Part III).

A significant feature of this book that distinguishes it from other textbooks on the subject is that Earth is assumed to be *ellipsoidal* in shape instead of spherical. Gravity can then vary meridionally as it does physically. This then provides the firm foundation for formulating more accurate, quantitative, atmospheric, and oceanic prediction models in the future. This book also aims to provide an easily accessible source for reference purposes. Part I can be considered to be a long book covering the fundamentals, and Parts II and III to be two shorter ones following on from Part I.

Themes I have endeavoured to convey the importance of:

- Generality.
- Scientific rigour.
- Dynamical and thermodynamical consistency.
- Unification of atmospheric and oceanic modelling.

Readership The only prerequisite to reading this book is a basic mastery of vector calculus and partial differentiation. This makes the book accessible to graduate students. Prior knowledge of fluid dynamics and thermodynamics is helpful, but not essential. How much detail a reader needs to understand a subject depends very much on their prior knowledge. I have deliberately erred on the side of giving more rather than less detail. I believe that students will appreciate this – insufficient detail impedes understanding – and that experienced readers can easily filter out details already familiar to them.

A thorough understanding of the governing equations for atmospheric and oceanic models and their properties is essential to the design of accurate and efficient numerical methods for their solution. This book is therefore intended to appeal not only to theoreticians but also to numerical modellers. It is in fact the book that I wished for when I first embarked on numerical modelling in the early 1970s, albeit some of the knowledge for this had not yet been developed.

Scope Although there is some (necessary) overlap of content, this book aims to complement other textbooks on atmospheric and oceanic fluid dynamics. One important way in which it does so is by representing gravity throughout in a more general and physically realistic manner.

Gravity is a dominant force in the geophysical-fluid-dynamical equations governing atmospheric and oceanic motion. It is mathematically specified in terms of the gradient of a potential, termed the geopotential. Textbooks generally assume from the very beginning that geopotential isosurfaces are spherical in shape. This is termed the spherical-geopotential approximation. Governing equations are then expressed in spherical-polar coordinates from that point on. This unduly restricts generality and also introduces inconsistencies into quantitative prediction systems.

For mildly oblate planets (such as Earth), geopotential surfaces are more realistically approximated as spheroids (i.e. slightly squashed spheres). In principle, this then allows gravity to vary from pole to equator (as it does in reality) – this is excluded by the spherical-geopotential approximation – and also to vary realistically in the vertical. Doing so in practice, however, is highly challenging. Based on recent advances, three chapters are therefore dedicated to:

1 The rigorous derivation from first principles (systematically developed in Chapters 7 and 8) of a physically realistic representation of geopotential (and therefore of gravity).
2 The development (in Chapter 12) of an associated, mutually consistent, *spheroidal*, coordinate system in terms of axial-orthogonal-curvilinear coordinates.

This then allows expression (in other chapters) of the governing equations in these more general (spheroidal) coordinates. The classical, spherical-geopotential approximation is a special case of this more general approach to the representation of gravity. As a side benefit, it can then be mathematically justified as a zeroth-order asymptotic approximation of the more general representation.

Expression of the governing equations in terms of axial-orthogonal-curvilinear coordinates instead of spherical-polar coordinates is slightly more complicated. It is, however, a small price to pay for maintaining generality of the governing equations. At any stage it is straightforward to express any equation in terms of spherical-polar coordinates by setting the metric factors to their spherical form. To aid clarity, to link to the literature, and for reference purposes, sets of governing equations are also given explicitly for the special case of spherical-polar coordinates.

Many textbooks focus on *maximum* simplification of the governing equations to better understand, in the simplest possible framework, basic physical processes. This also facilitates the development of accurate and efficient numerical methods for modelling them in practice. Here, the complementary approach of *minimum* simplification is taken instead to derive a *general* set of governing equations; just enough simplification of physical reality to be practically viable, but no more than is necessary. The goal is to ultimately improve the accuracy of quantitative atmospheric and oceanic models by using a more accurate set of governing equations.

Various approximated and mostly familiar equation sets are derived from this general set in later chapters, with a focus on preserving analogues of the fundamental conservation laws for mass, axial angular momentum, total energy, and potential vorticity. This is termed dynamical consistency. Variational methods provide a useful and complementary tool to examine this. An introduction to these methods is therefore given in Chapter 14.

The atmosphere and oceans exchange mass, momentum, and energy. Inconsistencies in the thermodynamics of atmospheric and oceanic models can lead to spurious climate drift in quantitative predictions. This is best avoided by specifying a Gibbs potential (or other thermodynamic potential), and obtaining *all* thermodynamic quantities from it. The thermodynamic framework for this important subject is developed in Chapter 9. This framework is essential for a physically realistic representation of ocean thermodynamics. As a side benefit, an acceptably realistic, analytically tractable representation of phase changes of water substance also is developed. This provides a useful introduction to more advanced representations.

Chapter Organisation The underpinning rationale for the ordering of chapters is given at the end of the Introduction (Chapter 1). Chapters can (almost) be read sequentially if one wishes. I say 'almost' because there are many interrelated facets involved in the formulation of governing equations for realistic atmospheric and oceanic forecast models. To accommodate this, and to understand how the many pieces of the formulational jigsaw fit together to form a complete picture, links are made in chapters to both earlier and later ones. These links may be ignored on a first reading.

Chapters do not, however, necessarily have to be read sequentially. Each chapter aims to be a 'scientific adventure', motivated by a stated goal, with a sign-posted journey to the destination. With a willingness to accept certain affirmations, or to refer to part of another chapter, each chapter can be read almost independently of the others. This is to help the reader interested in a particular subject to more readily access the relevant material without needing to first carefully read other chapters. This has led to some repetition, but this is no bad thing – it is how we acquire many skills in life.

Some chapters (particularly later ones) contain some original material specifically developed to answer various questions that arose during writing. This material is aimed at readers interested in exploring the current boundaries of research, particularly regarding the generality of governing equation sets, their exact solution for testing purposes, and their properties.

Course Examples Some suggestions on using this book for courses follow.
A basic course on:

- Formulation of Governing Equations for Atmospheric and Oceanic Models (or Dynamical Cores) – based on the chapters of Part I, with a flexible choice of material to be covered according to interest.

Specialist courses on:

- Representation of Gravity in Atmospheric and Oceanic Models – Chapters 7, 8, and 12.
- Introduction to Basic Thermodynamics for Atmospheric and Oceanic Dynamical Cores – Part of Chapter 4 plus Chapters 9–11, possibly supplemented by other material.
- Variational Methods and Hamilton's Principle of Stationary Action – Chapter 14.
- Conservation Principles of Atmospheric and Oceanic Dynamical Cores and Dynamical Consistency – Chapters 15–17.
- Shallow-Water Equations and the Non-divergent Barotropic Potential-Vorticity Equation – Chapters 18 and 19.
- Exact, Steady and Unsteady, Non-linear Solutions of Atmospheric and Oceanic Dynamical Cores – Chapters 20–23.

Notation Notation is for the most part 'standard'. However, the book draws on different disciplines of physics and mathematics. Each has its own conventions, and each generally makes full use of familiar Roman and Greek symbols, of which there are a limited number. This has inevitably led to some notational compromises. Within a subject area, I have mostly used its standard notation to facilitate access to its specialised literature, whilst identifying and clarifying potential notational ambiguity.

References To help readers more readily access related scientific literature, particularly for specialised aspects, I have provided more references than often found in textbooks. These references, nevertheless limited in both number and scope, are simply those that I am familiar with, that I have found helpful, that came to mind whilst writing this book and that were accessible to me after my retirement. They are intended to provide a portal to further references, of which there are many of great value.

Technical This book was written using the LyX Document Processor (www.lyx.org). LyX provides a comprehensive, user-friendly, graphical interface, with an integrated equation editor, to the underlying power and flexibility of the TeX/LaTeX typesetting system. Without this open-source software, writing this book would have been a far more daunting challenge. Many of the figures were created using the xfig (www.xfig.org) open-source software. I am grateful to Markus Gross for his valuable help to use LyX with TeX/LaTeX document classes, and also to use xfig. This got me off to a very good start.

Acknowledgements I am greatly indebted to my very good friend and former colleague, Andy White; his help and good humour have proven invaluable. Over our two decades of scientific collaboration, Andy has not only stimulated my interest in the subject matter of this book but also brought great clarity to it. He has carefully reviewed drafts of many chapters, identified various errors therein, and suggested many improvements. He also collaborated with me on various journal papers that have been adapted for use herein. Without Andy's help and encouragement over the years, particularly when morale was flagging, this book would never have been written!

I am also indebted to many other people, too numerous to list, with whom I have interacted over the past four decades or so. This includes former colleagues at Environment Canada and the UK Meteorological Office, national and international colleagues, and journal editors and reviewers. Collectively, they have all greatly contributed to and influenced my understanding of, and way of viewing, the formulation of governing equations for atmospheric and oceanic models. I am particularly grateful to two of my collaborators, John Thuburn and Nigel Wood. John carefully reviewed several chapters and, in particular, identified some misconceptions and the means to address them; and Chapter 13 extends collaborative work with Nigel. Various chapters in Part I of this book were much influenced by Geoff Vallis' excellent book entitled *Atmospheric and Oceanic Fluid Dynamics*. I am also indebted to Geoff for providing valuable advice on the book publication process and also for providing some LaTeX macros.

I am very grateful to: Matt Lloyd, Sarah Lambert, and Elle Ferns at CUP for their guidance and encouragement to a novice book author; Bret Workman for copyediting the manuscript; and Vidya Ashwin and her team at Integra Software Services for typesetting. They all greatly contributed to bringing a challenging project to fruition.

I would have liked to thank my former employer, the UK Meteorological Office, for its encouragement and support in writing this book. Sadly, I cannot since my invitations to do so were declined. C'est la vie, et on tourne la page; goodbye bureaucracy, hello retirement! Time will tell whether my early retirement years have been well spent …

I hope that readers of this book will find it both interesting and useful. At the very least, I found writing it to be challenging and fulfilling.

Finally, I thank my wife Lenore for her enduring love, patience, and support during the lengthy writing process.

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Notation and Acronyms

A partial list of the more important and frequently used symbols employed herein follows. They are grouped as Roman symbols, Greek symbols, and operators, and are organised within groups in roughly alphabetical order. A list of acronyms also is provided.

Notation is fairly standard, with relatively few exceptions. Vectors are generally written in bold, upright Roman, characters (as in $\mathbf{V}$); and scalars and indices in italic, Roman, and Greek ones. An overbar (as in $\overline{S}$) usually indicates a basic-state or horizontally averaged quantity; an overhat (as in $\widehat{S}$), a vertically averaged quantity; and a zero subscript (as in S_0), a representative value of a quantity.

Due to inevitable overlap of meaning between standard usage in different disciplines, some characters have different meanings in different chapters herein. In such instances, different meanings are separated by a semicolon (';') in their entry in the lists here, with the relevant meaning usually clear from the context in which the symbol appears.

Roman Symbols

a, c	Equatorial and polar radii, respectively, of axially symmetric spheroids/ellipsoids.
$\overline{a}$	Earth's mean radius.
$\mathbf{a}$	Label coordinates $\mathbf{a} = (a_1, a_2, a_3)$.
$\mathbf{A}, \mathbf{B}, \mathbf{C}$	Generic vectors in vector identities.
$A(R), C(R)$	Equatorial and polar radii, respectively, of geopotential surface $\Phi(\chi, R) = \Phi(\chi = 0, r = R) \equiv \Phi_R =$ constant.
$A(\xi_2), B(\xi_2)$	Two key functions in the derivation of the quasi-shallow equation set.
$\mathscr{A}$	Action of a physical system.
c_p, c_v	Specific heats at constant pressure and constant volume, respectively.
c_s	Local sound speed.
$\mathbb{C}$	A prototypical, materially conserved quantity for $\mathbb{C} = M, E, \Pi$.
ds, dS, dV	Distance metric, surface element, and volume element, respectively.
$đQ, đW, đC$	Infinitesimal changes in internal energy due to heat supplied, work done, and chemical composition, respectively.
e	First eccentricity $\left(a^2 - c^2\right)^{1/2}/a$, often shortened to eccentricity, of an ellipse or ellipsoid.
e'	Second eccentricity $\left(a^2 - c^2\right)^{1/2}/c$ of an ellipse or ellipsoid.
$(\mathbf{e}_1, \mathbf{e}_2, \mathbf{e}_3)$	Unit-vector triad in three, mutually orthogonal, curvilinear directions (ξ_1, ξ_2, ξ_3).
$E, \mathscr{E}$	Total and internal energies, respectively.
f	Coriolis parameter; a generic scalar function; Helmholtz energy.
$\mathbf{F}$	Force or sum of forces.
g	Gravity; Gibbs potential (or Gibbs free energy).
g_0	Value of 'standard gravity' for Earth ($g_0 \equiv 9.80665$ m s^{-2}).

g_S, g_S^E, g_S^P	Value of g at a planet's surface, equator, and a pole, respectively.
g^d, g^v, g^{av}	Gibbs potentials for dry air, water vapour, and humid air, respectively.
g^W, g^S	Gibbs potentials for pure liquid water and saline correction, respectively.
$g_S^P/g_S^E - 1$	Clairaut's fraction $\equiv \left(g_S^P - g_S^E\right)/g_S^E$.
$\mathbf{g}^N$	Newtonian gravitational force.
h	Elevation for geodetic coordinates.
h, h^0	Enthalpy and potential enthalpy, respectively.
$\left(h_1, h_2, h_3\right)$	Metric (or scale) factors for orthogonal-curvilinear coordinates (ξ_1, ξ_2, ξ_3).
H, B	Heights above a reference level of the top and bottom surfaces of a shallow layer of fluid, respectively.
$\widetilde{H}$	Depth of a shallow layer of fluid $\equiv H - B$.
$(\mathbf{i}_1, \mathbf{i}_2, \mathbf{i}_3)$	Unit-vector triad in three mutually orthogonal, Cartesian directions (x_1, x_2, x_3).
k	Integer index; wave number.
K	Kinetic energy.
$L, \mathscr{L}, \widehat{\mathscr{L}}$	Lagrangian, Lagrangian density, and vertically averaged Lagrangian density, respectively.
L_0^v, L_0^f	Latent heats of vaporisation and fusion, respectively.
$\mathbb{L}$	Latent energy.
m	Mixing ratio; $\Omega^2 a^3/(\gamma M)$, the ratio of centrifugal force to gravitational attraction; a point mass.
m, n	Order and degree of associated Legendre functions, respectively.
M, M_P	Total mass of a planet and its atmosphere (if present); axial absolute angular momentum.
M_E, M_A	Masses of Earth and of its atmosphere, respectively.
$\mathbb{M}$	Mass of an ideal gas or fluid parcel.
n_d	Number of independent degrees of freedom for a single molecule of an ideal gas.
$\mathbf{n}$	Unit vector normal to a surface.
$\mathbb{N}^D, \mathbb{N}^C, \mathbb{N}^P$	Number of independent degrees of freedom, components, and coexisting phases, respectively.
$\mathscr{N}$	Non-hydrostatic switch.
$p, \widehat{p}$	Pressure and vertically averaged pressure, respectively.
p^d, p^v	Partial pressures of dry air and water vapour, respectively.
p_{sat}^v	Saturation vapour pressure.
p_0	Constant reference pressure, usually set to 1 000 hPa or 1 1013.25 hPa.
p_0^d	Reference pressure for dry air.
p_0^{sat}	Saturation vapour pressure for pure water at $T = T_0$.
P_n	Legendre polynomial of degree n.
P_n^m, Q_n^m	Associated Legendre functions of the first and second kinds, respectively, of order m and degree n.
q	Mass fraction of water substance; $\ln p$.
$\dot{q}$	Rate of change of mass fraction.
$\dot{Q}, \dot{Q}_E$	Heating rate and rate of total-energy input, respectively.
Q	Quasi-hydrostatic switch.
$\mathbf{q}(t), \dot{\mathbf{q}}(t)$	Vector of generalised coordinates and its rate of change, respectively.
r	Radial distance from an origin (e.g. in spherical-polar coordinates).
$\mathtt{r}$	Radial distance measured perpendicular to a reference axis in cylindrical-polar coordinates.
$r_\perp$	Perpendicular distance from a rotation axis.
$\mathbf{r}$	Position vector.
R, R^d, R^v	Gas constants per unit mass of air, dry air, and water vapour, respectively.

$s, \dot{s}$	Generalised vertical coordinate and generalised vertical velocity, respectively.
S	Mass fraction of substance (e.g. of dry air, of water substance, or of salinity); a surface.
S_A	Mass fraction of Absolute Salinity.
S_P	Practical Salinity.
S_0	Constant reference value for Absolute Salinity.
$\dot{S}$	Rate of change of substance.
$\mathscr{S}$	Surface area of a planet.
t	Time.
$\mathbf{t}$	Unit vector tangent to a curve or surface.
T, T_v	Temperature and virtual temperature, respectively.
T_0	Temperature at the triple point of water for pure-water substance (in the absence of dry air); constant reference value for T.
T_*	Temperature at the triple point of water in the presence of dry air.
$\mathbf{u}$	Velocity vector in a uniformly rotating frame of reference.
$\mathbf{u}_{\text{hor}}$	Horizontal velocity.
V	Newtonian gravitational potential (i.e. the potential due solely to Newtonian gravity); potential energy in Newtonian particle mechanics; a volume.
V^C	Potential of the centrifugal force.
V_E	Volume of Earth.
$\mathbb{V}$	Volume of an ideal gas or a mixture of ideal gases.
$\mathscr{V}$	Volume of a planet.
$\dot{W}$	Rate of work done per unit mass.
X	Type of constituent.
(x_1, x_2, x_3)	Cartesian coordinates.
z	Altitude (for $z > 0$), $\lvert z \rvert$ is depth (for $z < 0$); axial position in cylindrical-polar coordinates.
$\widetilde{z}$	Basic terrain-following coordinate.
$\mathbf{Z}$	Absolute vorticity (i.e. the sum of planetary and relative vorticities).
$\mathbf{Z}_r, \mathbf{Z}_a$	Absolute vorticity in deep and shallow spherical-polar coordinates, respectively.

Greek Symbols

α	Specific volume.
β_T, β_T^*	First and second thermal-expansion coefficients, respectively.
$\widehat{\beta}^T, \widehat{\beta}^S$	Thermal-expansion and saline-contraction coefficients with respect to in situ temperature T, respectively.
$\beta_S, \beta_S^*, \beta_p$	Saline-contraction, heat-capacity, and compressibility coefficients, respectively.
γ	Ratio of specific heats, c_p/c_v; universal gravitational constant.
γM	Standard gravitational parameter for Earth + its atmosphere.
$\gamma M_E, \gamma M_A$	Standard gravitational parameters for Earth and its atmosphere, respectively.
γ^*	Thermobaric parameter.
Γ	Temperature lapse rate.
$\delta f, \Delta f$	Increments of a generic scalar f.
ε	Ellipticity $(a - c)/a$, sometimes called first flattening parameter; R^d/R^v.
$\widetilde{\varepsilon}$	$\left(a^2 - c^2\right) / \left(2c^2\right)$.
ζ	Relative vorticity.
η	Entropy.
θ	Potential temperature; parametric latitude.
Θ	Conservative temperature.
κ	R/c_p.

λ	Zonal (or azimuthal) coordinate in axially symmetric coordinate systems (e.g. in spherical-polar coordinates).
Λ	A materially conserved scalar field.
μ	Chemical potential in thermodynamics; reduced mass in particle mechanics; viscosity.
ν	Kinematic viscosity.
μ, ν	Two coefficients appearing in geopotential representation for GREAT coordinates.
(ξ_1, ξ_2, ξ_3)	Orthogonal-curvilinear coordinates in the $(\mathbf{e}_1, \mathbf{e}_2, \mathbf{e}_3)$ directions.
π	Ratio of a circle's circumference to its diameter; hydrostatic pressure.
$\Pi, \overline{\Pi}, \overline{\Pi}_{\xi_2}$	Potential vorticity, basic-state potential vorticity, and meridional gradient of $\overline{\Pi}$, respectively.
ρ, ρ^θ	Density and potential density, respectively.
σ	Pressure-based, terrain-following coordinate; frequency.
τ	Time, associated with label coordinates $\mathbf{a} = (a_1, a_2, a_3)$.
φ	Generic latitude.
ϕ	Geographic latitude.
Φ	Potential of apparent gravity, often termed geopotential.
χ	Geocentric latitude.
ψ	Stream function; a coordinate.
Ψ	Stream function; thermodynamic potential.
ω	Frequency.
$\mathbf{\Omega}, \widehat{\mathbf{\Omega}}$	Angular velocity (assumed constant), and unit vector aligned with its direction, respectively.

Operators

D/Dt	Material derivative in a rotating frame of reference.
$(D/Dt)_I$	Material derivative in an inertial frame of reference.
D_S/Dt	Material derivative in shallow (non-Euclidean) geometry, in a rotating frame of reference.
D_r/Dt	Material derivative in spherical-polar coordinates.
D_a/Dt	Material derivative in shallow (non-Euclidean) spherical-polar coordinates.
$\widehat{F}$	Vertical average of F.
$\mathfrak{L}$	A second-order differential operator associated with the non-divergent barotropic PV equation.
O	Order of a quantity.
$\delta F\left[\mathrm{q}(t)\right]$	Variation of functional $F\left[\mathrm{q}(t)\right]$.
∂	Partial-derivative operator.
$\partial(\mathbf{r})/\partial(\mathbf{a})$	Jacobian of the transformation from coordinates $\mathbf{r} = (x_1, x_2, x_3)$ to coordinates $\mathbf{a} = (a_1, a_2, a_3)$.
∇, ∇_S	Gradient operator in Euclidean and shallow non-Euclidean geometries, respectively.
∇_r, ∇_a	Gradient operator in deep and shallow spherical-polar coordinates, respectively.
∇_{hor}	Horizontal gradient operator.
∇^2, ∇_S^2	Laplacian operator in Euclidean and shallow non-Euclidean geometries, respectively.

Acronyms

BPV	Barotropic Potential Vorticity
BTF	Basic Terrain Following

COS	Confocal-Oblate-Spheroidal
GPS	Global Positioning System
GREAT	Geophysically Realistic, Ellipsoidal, Analytically Tractable
GSW	Gibbs Sea Water
IAPSO	International Association for the Physical Sciences of the Oceans
IAPWS	International Association for the Properties of Water and Steam
IOC	Intergovernmental Oceanographic Commission
PV	Potential Vorticity
SCOR	Scientific Committee on Oceanic Research
SIA	Seawater Ice Air
SOS	Similar-Oblate-Spheroidal
STF	Smooth Terrain Following
TEOS	Thermodynamic Equation Of Seawater
WGS	World Geodetic System