The General Theory of Relativity

The general theory of relativity, Einstein's theory of gravitation, has been included as a compulsory subject in undergraduate and graduate courses in Physics and Applied Mathematics all over the world. However, the physics-first approach that is taken by many textbooks is not universally used, as the approach often depends on the instructors' or students' background. Conceived from the lecture notes made by the author over a teaching career spanning 18 years, this book introduces the general theory of relativity for advanced students with a strong mathematical background.

The proposed book takes a 'math-first approach', for which the mathematical formalism comes first and is then applied to physics. It presents a concise yet comprehensive and structured understanding of the general theory of relativity. The book discusses the mathematical foundation of the general theory of relativity and focuses heavily on topics such as tensor calculus, geodesics, Einstein field equations, linearized gravity, Lie derivatives and their applications, the causal structure of spacetime, rotating black holes, and basic knowledge of cosmology and astrophysics. All of these are explained through a large number of worked examples and exercises.

Farook Rahaman is a Professor of Mathematics at Jadavpur University, Kolkata. Besides writing a book, *The Special Theory of Relativity*, he has published numerous research papers on galactic dark matter, wormhole geometry, charged fluid model, topological defects in the early universe, gravastars, black hole physics, star modeling, and the cosmological model of the universe.

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A Mathematical Approach

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To
my parents
Majeda Rahaman and Late Obaidur Rahaman
and
my son and wife
Md Rahil Miraj and Pakizah Yasmin

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Preface

At the beginning of the twentieth century, Einstein spent many years developing a new theory in physics. The newly developed theory is known as the theory of relativity. This is basically a combination of two theories: the first one is known as the special theory of relativity and latter one is dubbed as the general theory of relativity. The special theory of relativity is based on two postulates, namely the principle of relativity or equivalence, that is, the laws of physics are the same in all inertial systems, which means no preferred inertial system exists, while the second postulate is the principle of the constancy of the speed of light. The general theory of relativity asserts that there is no difference between the local effects of a gravitational field and that of acceleration of an inertial system. In other words, spacetime is warped or distorted by the matter and energy in it as an effect of gravity. According to the general theory of relativity, massive objects cause the outer space to twist due to gravity like a heavy ball bending a thin rubber sheet that is holding the ball. Heavier balls bend spacetime far more than lighter ones. Like the special theory of relativity, the general theory of relativity attracted scientists a lot, immediately after its discovery by Einstein. As a result, it has been included as a compulsory subject in graduate and postgraduate courses of physics and applied mathematics all over the globe. Einstein proposed the field equations for the general theory of relativity by applying his own intuition. Later, many other methods were developed to construct Einstein's field equations.

This book on the general theory of relativity is an outcome of a series of lectures delivered by me, over several years, to postgraduate students of mathematics at Jadavpur University. I should mention that it is not a fundamental book. This book has been written, from a mathematical point of view, after consulting several books existing in the literature. I have provided the list of the reference books. During my lectures, many students asked questions that helped me know their needs as well as the shortcomings in their understanding. Therefore, it is a well-planned textbook that has been organized in a logical order and every topic has been dealt with in a simple and lucid manner. A number of problems with hints, taken from the question papers of different universities, are included in each chapter.

The book is organized as follows:

In Chapter One a brief overview of tensor calculus, including the different types of tensors as well as operations on tensors, is given. Generalized Kronecker delta, Christoffel symbols, affine connection, covariant derivatives, geodesic coordinate, and various forms of tensors are described, with examples, as a foreground to understand the basics of general relativity. Chapter Two starts with a discussion of the geodesic equation in curved spacetime. In addition, several problems for different spacetimes are provided on geodesics. Chapter Three begins with the statement of three basic principles, namely Mach's principle, equivalence principle, and the principle of covariance. Next, the Einstein gravitational field equations are derived from the variational principle.

Also, in this chapter, the outline of some modified theories of gravity, such as f(R) theory of gravity, Gauss–Bonnet gravity, f(G) theory of gravity or modified Gauss–Bonnet gravity, f(T) theory of gravity, f(R,T) theory of gravity, Brans–Dicke theory of gravity, and Weyl gravity, are provided. A discussion on linearized gravity is given in Chapter Four. Newtonian limit of Einstein field equations or weak field approximation of Einstein field equations is derived. It is shown that Poisson's equation can be viewed as an approximation of Einstein field equations. A short mathematical description of gravitational wave is also provided. Chapter Five is dedicated to a short discussion on Lie derivatives and their applications. Killing equations and Killing vectors are also discussed with several examples. A short note on conformal Killing vector is also provided. Chapter Six is devoted to discussions on spacetimes of spherically symmetric distributions of matter. The exact exterior and interior solutions of Einstein field equations in spherically symmetric spacetimes are discussed. The proof of Birkoff's theory is provided. It states that a spherically symmetric gravitational field in vacuum is necessarily static and must have Schwarzschild form. The Tolman–Oppenheimer–Volkov (TOV) equation is discussed. Isotropic coordinate system is a new coordinate system whose spatial distance is proportional to the Euclidean square of the distances. Some static spherically symmetric spacetimes are rewritten in an isotropic coordinate system. A short discussion on interaction between the gravitational and electromagnetic fields are provided. Reissner–Nordström solution is a static solution of the gravitational field outside of a spherically symmetric charged body. Particle and photon orbits in the Schwarzschild spacetime are discussed in Chapter Seven. Also, in this chapter, using the trajectory in the gravitational field of sun (i.e., in the Schwarzschild spacetime), several tests of the theory of general relativity, namely the precession of the perihelion motion of mercury, bending of light, radar echo delay, and gravitational redshift, are explained. A discussion on the stable circular orbits in the Schwarzschild spacetime is given. A general treatment is provided for the experimental test of general theory of relativity for a general static and spherically symmetric configuration. Causal structure in the special theory of relativity, i.e., in Minkowski spacetime or flat spacetime, is characterized so that no massive particle can travel faster than light. In general relativity, locally there is no difference of the causality relation with Minkowski spacetime. However, globally, the causality relation is significantly different due to various spacetime topologies. A short discussion on causal structure of spacetimes is given in Chapter Eight. Several basic definitions and some standard theorems related to causality are explained. Chapter Nine deals with discussions on causal structures of specific spacetimes, which are the standard exact solutions of Einstein field equations such as Minkowski spacetime, de Sitter and anti-de Sitter spacetimes, Robertson–Walker spacetime, Bianchi-I spacetime, Schwarzschild spacetime, and Reissner–Nordström black hole. A short elementary discussion on rotating black holes is given in Chapter Ten. After introducing the tetrad, an outline of the derivation of the Kerr and Kerr–Newman solutions is illustrated through the complex transformation algorithm for both in four and higher dimensions. Some of the different forms of the Kerr solution are mentioned. Some elementary properties of the Kerr solution including the maximal extension of Kerr spacetime are discussed. Finally, brief discussions on Hawking radiation, Penrose process of extraction of energy from a Kerr black hole, and laws of black hole thermodynamics are given. Chapters Eleven and Twelve provide some simple applications of general theory of relativity in astrophysics and cosmology, respectively. Some preliminary concepts of extrinsic curvature, Lagrangian formalism of the general theory of relativity, and $3 + 1$ decomposition of spacetime are given as appendices.

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