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Excerpt
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CHAPTER 1 Tensor Calculus — A Brief Overview

1.1 Introduction

The principal target of tensor calculus is to investigate the relations that remain the same when we change from one coordinate system to any other. The laws of physics are independent of the frame of references in which physicists describe physical phenomena by means of laws. Therefore, it is useful to exploit tensor calculus as the mathematical tool in which such laws can be formulated.

1.2 Transformation of Coordinates

Let there be two reference systems, S with coordinates $(x^1, x^2, \ldots, x^n)$ and $\overline{S}$ with coordinates $(\overline{x}^1, \overline{x}^2, \ldots, \overline{x}^n)$ (Fig. 1). The new system $\overline{S}$ depends on the old system S as

$$\overline{x}^i = \phi^i(x^1, x^2, \ldots, x^n); \quad i = 1, 2, \ldots, n. \tag{1.1}$$

Here ϕ^i are single-valued continuous differentiable functions of $x^1, x^2, \ldots, x^n$ and further the Jacobian

$$\left|\frac{\partial \phi^i}{\partial x^j}\right| = \begin{vmatrix} \frac{\partial \phi^1}{\partial x^1} & \frac{\partial \phi^1}{\partial x^2} & \frac{\partial \phi^1}{\partial x^3} & \cdots & \frac{\partial \phi^1}{\partial x^n} \\ \frac{\partial \phi^2}{\partial x^1} & \frac{\partial \phi^2}{\partial x^2} & \frac{\partial \phi^2}{\partial x^3} & \cdots & \frac{\partial \phi^2}{\partial x^n} \\ \cdots & \cdots & \cdots & \cdots & \cdots \\ \frac{\partial \phi^n}{\partial x^1} & \frac{\partial \phi^n}{\partial x^2} & \frac{\partial \phi^n}{\partial x^3} & \cdots & \frac{\partial \phi^n}{\partial x^n} \end{vmatrix} \neq 0.$$

Differentiation of Eq. (1.1) yields

$$d\overline{x}^i = \sum_{r=1}^{n} \frac{\partial \phi^i}{\partial x^r} dx^r = \sum_{r=1}^{n} \frac{\partial \overline{x}^i}{\partial x^r} dx^r = \sum_{r=1}^{n} \overline{a}^i_r dx^r.$$

Now and onward, we use the Einstein summation convention, i.e., omit the summation symbol $\sum$ and write the above equations as

$$d\overline{x}^i = \frac{\partial \overline{x}^i}{\partial x^r} dx^r = \overline{a}^i_r dx^r, \tag{1.2}$$

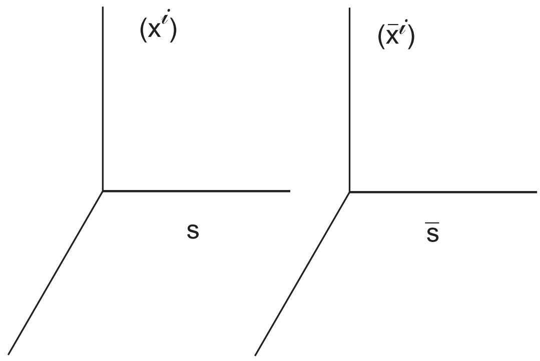

Figure 1 S and $\bar{S}$ frames.

or

$$dx^i = \frac{\partial x^i}{\partial \bar{x}^m} d\bar{x}^m = a^i_m d\bar{x}^m. \tag{1.3}$$

The repeated index r or m is known as **dummy index**. The index i is not dummy and is known as **free index**.

The transformation matrices are inverse to each other

$$\bar{a}^i_r \, a^m_i = \delta^m_r. \tag{1.4}$$

The symbol δ^m_r is Kronecker delta, is defined as

$$\delta^m_r = 1 \quad if \quad m = r$$
$$= 0 \quad if \quad m \neq r$$

Obviously vectors in $(\bar{S})$ system are linked with (S) system.

1.3 Covariant and Contravariant Vector and Tensor

Usually one can describe the tensors by means of their properties of transformation under coordinate transformation. There are two possible ways of transformations from one coordinate system (x^i) to the other coordinate system $(\bar{x}^i)$.

Let us consider a set of n functions A_i of the coordinates x^i. The functions A_i are said to be the components of **covariant vector** if these components transform according to the following rule

$$\bar{A}_i = \frac{\partial x^j}{\partial \bar{x}^i} A_j. \tag{1.5}$$

Also, one can find by multiplying $\frac{\partial \bar{x}^i}{\partial x^k}$ and using $\frac{\partial \bar{x}^i}{\partial x^k}\frac{\partial x^j}{\partial \bar{x}^i} = \delta^j_k$ and $\delta^j_k A_j = A_k$

$$A_k = \frac{\partial \bar{x}^i}{\partial x^k} \bar{A}_i.$$

Exercise 1.1

Gradient of a scalar B, i.e., $B_i = \frac{\partial B}{\partial x_i}$ is a covariant vector.

Here, A_i is known as the **covariant tensor of first order or of the type** $(0, 1)$**.**

The functions A^i are said to be the components of the **contravariant vector** if these components transform according to the following rule

$$\bar{A}^i = \frac{\partial \bar{x}^i}{\partial x^j} A^j \tag{1.6}$$

Also, one can find by multiplying both sides with $\frac{\partial x^k}{\partial \bar{x}^i}$ and using $\delta^k_j A^j = A^k$

$$A^k = \frac{\partial x^k}{\partial \bar{x}^i} \bar{A}^i.$$

Here, A^i is known as the **contravariant tensor of first order or of the type** $(1, 0)$**.**

Exercise 1.2

Tangent vector $\frac{dx^i}{du}$ of the curve $x^i = x^i(u)$ is a contravariant vector.

Exercise 1.3

Let components of velocity vector in Cartesian coordinates are $\dot{x}$ and $\dot{y}$. Find corresponding components in polar coordinates.
Hint: Here, $x^1 = x$, $x^2 = y$, and $\overline{x}^1 = r, \overline{x}^2 = \theta$ with $x = r\cos\theta$, $y = r\sin\theta$, i.e., $r = \sqrt{x^2 + y^2}$, $\theta = \tan^{-1}(\frac{y}{x})$.
Let $A^1 = \dot{x}$, $A^2 = \dot{y}$. We will have to find $\overline{A}^1, \overline{A}^2$.
("dot" denotes differentiation with respect to t.)
Using the definition $\overline{A}^i = \frac{\partial \bar{x}^i}{\partial x^j} A^j$, we have

$$\overline{A}^1 = \frac{\partial \bar{x}^1}{\partial x^1} A^1 + \frac{\partial \bar{x}^1}{\partial x^2} A^2 \ or, \ \overline{A}^1 = \frac{\partial r}{\partial x}\dot{x} + \frac{\partial r}{\partial y}\dot{y} = \dot{r}.$$

Similarly,

$$\overline{A}^2 = \frac{\partial \theta}{\partial x}\dot{x} + \frac{\partial \theta}{\partial y}\dot{y} = \dot{\theta}.$$

Exercise 1.4

Let components of acceleration vector in Cartesian coordinates be $\ddot{x}$ and $\ddot{y}$. Find corresponding components in polar coordinates.

Hint: Let $A^1 = \ddot{x}$, $A^2 = \ddot{y}$. We will have to find $\overline{A}^1, \overline{A}^2$.
Here,

$$\overline{A}^1 = \frac{\partial r}{\partial x}\ddot{x} + \frac{\partial r}{\partial y}\ddot{y} = \ddot{r} - r\dot{\theta},\ \overline{A}^2 = \frac{\partial \theta}{\partial x}\ddot{x} + \frac{\partial \theta}{\partial y}\ddot{y} = \ddot{\theta} + \frac{2}{r}\dot{\theta}\dot{r}.$$

1.3.1 Invariant

Let ϕ be a function of coordinate system (x^i) and $\overline{\phi}$ be its transform in another coordinate system $(\overline{x}^i)$. Then, ϕ is said to be **invariant** if $\overline{\phi} = \phi$.

Exercise 1.5

The expression A^iB_i is an invariant or scalar, i.e.,

$$\bar{A}^i\bar{B}_i = A^iB_i. \tag{1.7}$$

Hint: Use definitions given in Eqs. (1.5) and (1.6).

An **invariant or scalar** is known as the **tensor of the type** $(0, 0)$**.**

1.3.2 Contravariant and covariant tensors of rank two

Let C^i and B^j be two contravariant vectors with n components, then $C^iB^j = A^{ij}$ has n^2 quantities, i.e., A^{ij} are the set of n^2 functions of the coordinates x^i. If the transformation of A^{ij} is like

$$\bar{A}^{ij} = \frac{\partial \bar{x}^i}{\partial x^k}\frac{\partial \bar{x}^j}{\partial x^l}A^{kl}, \tag{1.8}$$

then A^{ij} is known as **contravariant tensor of rank two**. Here, A^{ij} is also known as the **contravariant tensor of order two or of the type** $(2, 0)$**.**

If we multiply both sides of (1.8) by $\frac{\partial x^r}{\partial \bar{x}^i}\frac{\partial x^s}{\partial \bar{x}^j}$, then

$$A^{rs} = \frac{\partial x^r}{\partial \bar{x}^i}\frac{\partial x^s}{\partial \bar{x}^j}\bar{A}^{ij}.$$

Again, if C_i and B_j are two covariant vectors with n components, then $C_iB_j = A_{ij}$ form n^2 quantities, i.e., A_{ij} are the set of n^2 functions of the coordinates x^i.

If the transformation of A_{ij} is like

$$\bar{A}_{ij} = \frac{\partial x^k}{\partial \bar{x}^i}\frac{\partial x^l}{\partial \bar{x}^j}A_{kl}, \tag{1.9}$$

then A_{ij} is known as **covariant tensor of rank two**.

Here, A_{ij} is also known as the **covariant tensor of order two or of the type** $(0, 2)$**.**

If we multiply both sides of (1.9) by $\frac{\partial \bar{x}^i}{\partial x^r}\frac{\partial \bar{x}^j}{\partial x^s}$, then

$$A_{rs} = \frac{\partial \bar{x}^i}{\partial x^r}\frac{\partial \bar{x}^j}{\partial x^s}\bar{A}_{ij}.$$

1.3.3 Mixed tensor of order two A^i_j

Suppose A^i_j is a set of n^2 functions of n coordinates. If the transformation obeys the following rule

$$\bar{A}^i_j = \frac{\partial \bar{x}^i}{\partial x^k}\frac{\partial x^l}{\partial \bar{x}^j}A^k_l,$$

then A^k_l is known as the **mixed tensor of order two or of the type** (1, 1)**.**

Thus, mixed tensor of order two can be obtained by taking a covariant vector A_i and a contravariant vector B^j, i.e., $C^j_i = A_i B^j$.

Exercise 1.6

Kronecker delta δ^j_i is a mixed tensor of order two.
Hint: If δ^j_i can be combined with components of two vectors to form a scalar, then δ^j_i will be a tensor. Now

$$A^i B_j \delta^j_i = A^i B_i = scalar.$$

If the transformation obeys the following rule

$$\bar{A}^{i_1 i_2 \ldots i_p}_{j_1 j_2 \ldots j_q} = \frac{\partial \bar{x}^{i_1}}{\partial x^{k_1}}\frac{\partial \bar{x}^{i_2}}{\partial x^{k_2}}\cdots\frac{\partial \bar{x}^{i_p}}{\partial x^{k_p}}\frac{\partial x^{l_1}}{\partial \bar{x}^{j_1}}\frac{\partial x^{l_2}}{\partial \bar{x}^{j_2}}\cdots\frac{\partial x^{l_q}}{\partial \bar{x}^{j_q}}A^{k_1 k_2 \ldots k_p}_{l_1 l_2 \ldots l_q},$$

then $A^{k_1 k_2 \ldots k_p}_{l_1 l_2 \ldots l_q}$ is known as **mixed tensor of the type** (p, q)**.**

1.3.4 Symmetric and skew-symmetric tensors

If a tensor is unaltered after changing every pair of contravariant or covariant indices, then it is said to be a symmetric tensor. Let $T_{\alpha\beta}$ be a covariant tensor of rank two.

If $T_{\alpha\beta} = T_{\beta\alpha}$, *then it is known as* **symmetric tensor**.

If a tensor is altered in its sign but not in magnitude after changing every pair of contravariant or covariant indices, then it is said to be a skew-symmetric tensor.

If $T_{\alpha\beta} = -T_{\beta\alpha}$, *then it is known as* **antisymmetric or skew-symmetric tensor**.

Exercise 1.7

Kronecker delta δ_{ij} is a symmetric tensor.

Exercise 1.8

If A_i is covariant vector, then $curlA_i = \frac{\partial A_i}{\partial x_j} - \frac{\partial A_j}{\partial x_i}$ is a skew-symmetric tensor.

Hint: Use $curlA_i = \frac{\partial A_i}{\partial x_j} - \frac{\partial A_j}{\partial x_i} = B_{ij}$ and show that $B_{ij} = -B_{ji}$.

Note 1.1

Symmetry property of a tensor is independent of the coordinate system.

Note 1.2

A symmetric tensor of order two in n-dimensional space has at most $\frac{n(n+1)}{2}$ independent components whereas an antisymmetric tensor of order two has at most $\frac{n(n-1)}{2}$ independent components.

1.4 Operations on Tensors

i. The addition and subtraction of two tensors of the same type is a tensor of same type.

Exercise 1.9

$$A_{ij} \pm B_{ij} = C_{ij},\ A^{ij} \pm B^{ij} = C^{ij},\ A_i^j \pm B_i^j = C_i^j$$

Exercise 1.10

Any covariant or contravariant tensor of second order can be expressed as a sum of a symmetric and a skew-symmetric tensor of order two.
Hint:

$$a_{ij} = \frac{1}{2}(a_{ij} + a_{ji}) + \frac{1}{2}(a_{ij} - a_{ji}), \quad \text{etc.}$$

ii. The type of the tensor remains invariant by multiplication of a scalar α.

Exercise 1.11

$$\alpha A_{ij} = C_{ij}, \quad \alpha A^{ij} = C^{ij}, \quad \alpha A_i^j = C_i^j$$

iii. **Outer product:** The outer product of two tensors is a new tensor whose order is the sum of the orders of the given tensors.

Exercise 1.12

Let two tensors of types (2,3) and (1,2) be respectively, A^{ij}_{klm} and B^a_{bc}, then the outer product of these tensors has type (3,5), i.e.,

$$A^{ij}_{klm} B^a_{bc} = T^{ija}_{klmbc}$$

iv. **Contraction:** The particular type of operation by which the order (r) of a mixed tensor is lowered by order ($r - 2$) is known as contraction.

Exercise 1.13

Let A^{ij}_{klm} be a mixed tensor of order five. The new tensor A^{ij}_{kim} can be obtained by replacing lower index l by the upper index i and taking summation over i, one gets the tensor of order three.

$$A^{ij}_{kim} = B^j_{km}$$

v. **Inner product:** The outer product of two tensors followed by contraction with respect to an upper index and a lower index of the other results in a new tensor which is called an inner product.

Exercise 1.14

$$A^{ij}_{k} B^{k}_{mn} \equiv C^{ijk}_{kmn} = D^{ij}_{mn}, \quad A^{ij}_{k} B^{m}_{ij} = D^{m}_{k}$$

1.4.1 Test for tensor character: Quotient Law

An entity whose inner product by an arbitrary tensor (covariant or contravariant) always gives a tensor is itself a tensor.

Exercise 1.15

If $C(i,j)A^iB^j$ is an invariant, then $C(i,j) = C_{ij}$ is a tensor of the type (0,2).

Exercise 1.16

If $C(p,q,r)B^{qs}_r = A^s_p$, then $C(p,q,r) = C^r_{pq}$ is a tensor of the type (1,2).

Exercise 1.17

Let λ^i, μ^i be the components of two arbitrary vectors with $a_{hijk}\lambda^h\mu^i\lambda^j\mu^k = 0$, then prove that

$$a_{hijk} + a_{hkji} + a_{jihk} + a_{jkhi} = 0.$$

Hint: Given that

$$A = a_{hijk}\lambda^h\mu^i\lambda^j\mu^k = 0.$$

Differentiating with respect to λ^h, we get

$$\frac{\partial A}{\partial \lambda^h} = a_{hijk}\mu^i\lambda^j\mu^k + a_{pihk}\lambda^p\mu^i\mu^k = 0.$$

Again, differentiating with respect to λ^j, we get

$$\frac{\partial^2 A}{\partial \lambda^h \partial \lambda^j} = a_{hijk}\mu^i\mu^k + a_{jihk}\mu^i\mu^k = 0.$$

Now, differentiating with respect to μ^i and μ^k, one will find, respectively,

$$\frac{\partial^3 A}{\partial \lambda^h \partial \lambda^j \partial \mu^i} = a_{hijk}\mu^k + a_{hkji}\mu^k + a_{jihk}\mu^k + a_{jkhi}\mu^k = 0,$$
$$\frac{\partial^4 A}{\partial \lambda^h \partial \lambda^j \partial \mu^i \partial \mu^k} = a_{hijk} + a_{hkji} + a_{jihk} + a_{jkhi} = 0.$$

Exercise 1.18

If A^i is an arbitrary contravariant vector and $C_{ij}A^iA^j$ is an invariant, then show that $C_{ij} + C_{ji}$ is a covariant tensors of the second order.
Hint: Given $C_{ij}A^iA^j$ is an invariant for arbitrary contravariant vector A^i, therefore,

$$C_{ij}A^iA^j = C'_{ij}A'^iA'^j.$$

Tensor law of transformation yields

$$C_{ij}A^iA^j = C'_{ij}\frac{\partial x'^i}{\partial x^\alpha}A^\alpha\frac{\partial x'^j}{\partial x^\beta}A^\beta.$$

Now interchanging the suffix i and j

$$C_{ji}A^jA^i = C'_{ji}\frac{\partial x'^j}{\partial x^\alpha}\frac{\partial x'^i}{\partial x^\beta}A^\alpha A^\beta = C'_{ji}\frac{\partial x'^i}{\partial x^\alpha}\frac{\partial x'^j}{\partial x^\beta}A^\alpha A^\beta.$$

(interchanging the dummy suffixes α and β)
Thus,

$$(C_{ji} + C_{ij})A^iA^j = (C'_{ji} + C'_{ij})\frac{\partial x'^i}{\partial x^\alpha}\frac{\partial x'^j}{\partial x^\beta}A^\alpha A^\beta,$$
$$\Rightarrow (C_{\alpha\beta} + C_{\beta\alpha})A^\alpha A^\beta = (C'_{ji} + C'_{ij})\frac{\partial x'^i}{\partial x^\alpha}\frac{\partial x'^j}{\partial x^\beta}A^\alpha A^\beta,$$
$$\Rightarrow \left[(C_{\alpha\beta} + C_{\beta\alpha}) - (C'_{ij} + C'_{ji})\frac{\partial x'^i}{\partial x^\alpha}\frac{\partial x'^j}{\partial x^\beta}\right]A^\alpha A^\beta = 0.$$

Since A^α is arbitrary, therefore, the expression within the square bracket vanishes. Hence, $C_{\alpha\beta} + C_{\beta\alpha}$ is a $(0, 2)$-tensor.

1.4.2 Conjugate or reciprocal tensor of a tensor

Consider a symmetric covariant tensor of second order a_{ij}, i.e., of the type (0,2) whose determinant, $|a_{ij}|$ is nonzero; then

$$b^{ij} = \frac{cofactor\ of\ a_{ij}\ in\ |a_{ij}|}{|a_{ij}|}$$

is known as reciprocal tensor of a_{ij}. It is of the type (2,0).

Note 1.3

Reciprocal tensor exists for any tensor. Only condition being its determinant is nonzero. Here, $a_{ij}b^{ik} = \delta^k_j$ and $|a_{ij}||b^{ik}| = |\delta^k_j| = 1$. Usually, conjugate of a_{ij} is written as a^{ij} and $a_{ij}a^{ij} = \delta^j_j = n$.

Note 1.4

Tensor equations in one system (x^i) remain valid in all other coordinate systems $(\overline{x}^i)$, e.g., if $T^i_{jkl} = 2T^i_{ljk}$, then $\overline{T}^i_{jkl} = 2\overline{T}^i_{ljk}$.

1.5 Generalized Kronecker Delta

The generalized Kronecker Delta $\delta^{\alpha\beta}_{\mu\nu}$ is defined as follows:

$$\begin{aligned}\delta^{\alpha\beta}_{\mu\nu} &= \begin{vmatrix}\delta^\alpha_\mu & \delta^\beta_\mu \\ \delta^\alpha_\nu & \delta^\beta_\nu\end{vmatrix} \\ &= +1,\ \alpha \neq \beta,\ \alpha = \mu,\ \beta = \nu \\ &= -1,\ \alpha \neq \beta,\ \alpha = \nu,\ \beta = \mu \\ &= 0,\ otherwise.\end{aligned}$$

We can define $\delta^{\alpha\beta\gamma}_{\mu\nu\xi}$ and $\delta^{\alpha\beta\gamma\rho}_{\mu\nu\xi\omega}$ as follows:

$$\delta^{\alpha\beta\gamma}_{\mu\nu\xi} = \begin{vmatrix}\delta^\alpha_\mu & \delta^\beta_\mu & \delta^\gamma_\mu \\ \delta^\alpha_\nu & \delta^\beta_\nu & \delta^\gamma_\nu \\ \delta^\alpha_\xi & \delta^\beta_\xi & \delta^\gamma_\xi\end{vmatrix},$$

$$\delta^{\alpha\beta\gamma\rho}_{\mu\nu\xi\omega} = \begin{vmatrix}\delta^\alpha_\mu & \delta^\beta_\mu & \delta^\gamma_\mu & \delta^\rho_\mu \\ \delta^\alpha_\nu & \delta^\beta_\nu & \delta^\gamma_\nu & \delta^\rho_\nu \\ \delta^\alpha_\xi & \delta^\beta_\xi & \delta^\gamma_\xi & \delta^\rho_\xi \\ \delta^\alpha_\omega & \delta^\beta_\omega & \delta^\gamma_\omega & \delta^\rho_\omega\end{vmatrix}.$$

Exercise 1.19

$$\delta^{123}_{123} = \delta^{123}_{231} = 1,$$
$$\delta^{123}_{213} = \delta^{123}_{132} = -1.$$

Exercise 1.20

Show that

$$\delta^{\alpha\beta}_{\mu\beta} = 3\delta^{\alpha}_{\mu}.$$

Exercise 1.21

Show that

$$\delta^{\alpha}_{\alpha} = 4.$$

Exercise 1.22

Show that

$$\delta^{\alpha\beta\tau}_{\mu\gamma\tau} = 2\delta^{\alpha\beta}_{\mu\gamma}.$$

Hint:

$$\delta^{\alpha\beta\tau}_{\mu\gamma\tau} = \begin{vmatrix} \delta^{\alpha}_{\mu} & \delta^{\beta}_{\mu} & \delta^{\tau}_{\mu} \\ \delta^{\alpha}_{\gamma} & \delta^{\beta}_{\gamma} & \delta^{\tau}_{\gamma} \\ \delta^{\alpha}_{\tau} & \delta^{\beta}_{\tau} & \delta^{\tau}_{\tau} \end{vmatrix}.$$

Now, expand along third row and use $\delta^{\tau}_{\tau} = 4$

Exercise 1.23

Show that

$$\delta^{\alpha\beta\tau\rho}_{\mu\nu\gamma\rho} = -\begin{vmatrix} \delta^{\alpha}_{\mu} & \delta^{\beta}_{\mu} & \delta^{\tau}_{\mu} \\ \delta^{\alpha}_{\nu} & \delta^{\beta}_{\nu} & \delta^{\tau}_{\nu} \\ \delta^{\alpha}_{\gamma} & \delta^{\beta}_{\gamma} & \delta^{\tau}_{\gamma} \end{vmatrix}.$$