Matrix Mathematics

Using a modern matrix-based approach, this rigorous second course in linear algebra helps upper-level undergraduates in mathematics, data science, and the physical sciences transition from basic theory to advanced topics and applications. Its clarity of exposition together with many illustrations, 900+ exercises, and 350 conceptual and numerical examples aid the student's understanding. Concise chapters promote a focused progression through essential ideas. Topics are derived and discussed in detail, including the singular value decomposition, Jordan canonical form, spectral theorem, *QR* factorization, normal matrices, Hermitian matrices, and positive definite matrices. Each chapter ends with a bullet list summarizing important concepts. New to this edition are chapters on matrix norms and positive matrices, many new sections on topics including interpolation and *LU* factorization, 300+ additional problems, many new examples, and color-enhanced figures. Prerequisites include a first course in linear algebra and basic calculus sequence. Instructor resources are available.

Stephan Ramon Garcia is W. M. Keck Distinguished Service Professor and Chair of the Department of Mathematics and Statistics at Pomona College. He is the author of five books and over 100 research articles in operator theory, complex analysis, matrix analysis, number theory, discrete geometry, and combinatorics. He has served on the editorial boards of the *Proceedings of the American Mathematical Society, Notices of the American Mathematical Society, Involve,* and *The American Mathematical Monthly.* He received six teaching awards from three different institutions and is a fellow of the American Mathematical Society, which has awarded him the inaugural Dolciani Prize for Excellence in Research.

Roger A. Horn was Professor and Chair of the Department of Mathematical Sciences at the Johns Hopkins University and then Research Professor of Mathematics at the University of Utah until his retirement in 2015. His publications include *Matrix Analysis* (2nd edition, Cambridge, 2012) and *Topics in Matrix Analysis* (both written with Charles R. Johnson, Cambridge, 1991), as well as more than 100 research articles in matrix analysis, statistics, health services research, complex variables, probability, differential geometry, and analytic number theory. He was the editor of *The American Mathematical Monthly* and has served on the editorial boards of the *SIAM Journal of Matrix Analysis, Linear Algebra and Its Applications,* and the *Electronic Journal of Linear Algebra.*

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Matrix Mathematics

A Second Course in Linear Algebra

SECOND EDITION

STEPHAN RAMON GARCIA
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To our families:

Gizem, Reyhan, and Altay
Susan;
Craig, Cori, Cole, and Carson;
Howard, Heidi, Archer, and Ella Ceres

"A broad coverage of more advanced topics, rich set of exercises, and thorough index make this stylish book an excellent choice for a second course in linear algebra."

Nick Higham, *University of Manchester*

"This textbook thoroughly covers all the material you'd expect in a Linear Algebra course plus modern methods and applications. These include topics like the Fourier transform, eigenvalue adjustments, stochastic matrices, interlacing, power method and more. With 20 chapters of such material, this text would be great for a multi-part course and a reference book that all mathematicians should have."

Deanna Needell, *University of California, Los Angeles*

"The original edition of Garcia and Horn's *Second Course in Linear Algebra* was well-written, well-organized, and contained several interesting topics that students should see – but rarely do in first-semester linear algebra – such as the singular value decomposition, Gershgorin circles, Cauchy's interlacing theorem, and Sylvester's inertia theorem. This new edition also has all of this, together with useful new material on matrix norms. Any student with the opportunity to take a second course on linear algebra would be lucky to have this book."

Craig Larson, *Virginia Commonwealth University*

"An extremely versatile Linear Algebra textbook that allows numerous combinations of topics for a traditional course or a more modern and applications-oriented class. Each chapter contains the exact amount of information, presented in a very easy-to-read style, and a plethora of interesting exercises to help the students deepen their knowledge and understanding of the material."

Maria Isabel Bueno Cachadina, *University of California, Santa Barbara*

"This is an excellent textbook. The topics flow nicely from one chapter to the next and the explanations are very clearly presented. The material can be used for a good second course in Linear Algebra by appropriately choosing the chapters to use. Several options are possible. The breadth of subjects presented makes this book a valuable resource."

Daniel B. Szyld, *Temple University and President of the International Linear Algebra Society*

"With a careful selection of topics and a deft balance between theory and applications, the authors have created a perfect textbook for a second course on Linear Algebra. The exposition is clear and lively. Rigorous proofs are supplemented by a rich variety of examples, figures, and problems."

Rajendra Bhatia, *Ashoka University*

"The authors have provided a contemporary, methodical, and clear approach to a broad and comprehensive collection of core topics in matrix theory. They include a wealth of illustrative examples and accompanying exercises to re-enforce the concepts in each chapter. One unique aspect of this book is the inclusion of a large number of concepts that arise in many interesting applications that do not typically appear in other books. I expect this text will be a compelling reference for active researchers and instructors in this subject area."

Shaun Fallat, *University of Regina*

"It starts from scratch, but manages to cover an amazing variety of topics, of which quite a few cannot be found in standard textbooks. All matrices in the book are over complex numbers, and the connections to physics, statistics, and engineering are regularly highlighted. Compared with the first edition, two new chapters and 300 new problems have been added, as well as many new conceptual examples. Altogether, this is a truly impressive book."

Claus Scheiderer, *University of Konstanz*

Contents

Preface for the Second Edition

New to this Edition

This is the second edition of *A Second Course in Linear Algebra*. The new title reflects an approach to advanced linear algebra that emphasizes matrix factorizations and algorithms. The second edition includes:

- New chapters on Matrix Norms and Positive Matrices.
- Revisions that incorporate classroom experience with students.
- New sections on Interpolation, Orthogonal polynomials, Gaussian quadrature, *LU* factorization, unitary equivalence and bidiagonal matrices, induced matrix norms, iterative algorithms such as the power and point Jacobi methods, and Perron-Frobenius theory.
- Color-enhanced figures.
- More than 300 new problems and many new conceptual and numerical examples.
- A comprehensive solution manual available to instructors.

Target Readership

Matrix mathematics and linear algebra are increasingly relevant in a world focused on the acquisition and analysis of data. Consequently, this book is intended for students of pure and applied mathematics, computer science, economics, engineering, mathematical biology, operations research, physics, and statistics. We assume that the reader has completed a lower-division calculus sequence and a first course in linear algebra. Analysis is not a prerequisite for this book.

Key Features of the Book

- Block matrices are employed systematically.
- Matrix factorizations and unitary transformations are emphasized.
- More than 350 examples illustrate concepts introduced in the text.

- Topics for a one-semester course can be selected in many ways to match the needs and interests of the class.
- Reviews of complex numbers, polynomials, and basic linear algebra are included.
- More than 90 figures illustrate the geometric foundations of linear algebra.
- Special topics include polynomial interpolation, orthogonal polynomials, Gaussian quadrature, matrix norms, Perron–Frobenius theory, and the Google matrix.
- Every chapter includes problems, more than 900 in total.
- Notes at the end of chapters provide sources of additional information.
- Each chapter ends with a bullet list of important concepts.
- Symbols used in the book are listed in a table of notation, with page references.
- A comprehensive index helps readers locate concepts and definitions. More than 2,000 entries enhance the value of the book as a reference.
- Concise, direct presentation and language level are suitable for an international audience.

Coverage of the Book

Matrices and vector spaces in this book are over the complex field. The use of complex scalars is essential to the study of eigenvalues, even for real matrices, and is consistent with modern numerical linear algebra software. Moreover, it is aligned with applications in physics (complex wave functions and Hermitian matrices in quantum mechanics), electrical engineering (analysis of circuits and signals in which both phase and amplitude are important), statistics (time series and characteristic functions), and computer science (fast Fourier transforms, convergent matrices in iterative algorithms, and quantum computing).

While studying linear algebra with this book, students can observe and practice good mathematical communication skills. These skills include how to state (and read) a theorem carefully; how to choose (and use) hypotheses; how to prove a statement by induction, by contradiction, or by contraposition; how to improve a theorem by weakening its hypotheses or strengthening its conclusions; how to use counterexamples; and how to write a cogent solution to a problem.

The following topics in the book are useful in applications of linear algebra, but fall outside the realm of linear transformations and similarity, so they may be absent from textbooks that adopt an abstract operator approach:

- Gershgorin's disk theorem on eigenvalue location
- The pivot-column decomposition and full-rank factorizations
- Commutants and trace-zero matrices (Shoda's theorem)
- *QR*, bidiagonal, triangular, and Cholesky factorizations
- Discrete Fourier transforms
- Circulant matrices
- Eigenvalue adjustments and the Google matrix
- Nonnegative matrices (Markov matrices) and positive matrices (Perron's theorem)

- The singular value and compact singular value decompositions
- Low-rank approximations to a data matrix
- Generalized inverses (Moore–Penrose inverses)
- Positive semidefinite matrices
- Schur complements
- Hadamard (entrywise) and Kronecker (tensor) products
- The Schur product theorem
- Matrix norms and the spectral radius
- Error bounds for eigenvalue computations (Bauer–Fike theorem)
- Convergent matrices, power-bounded matrices, and iterative algorithms
- Least-squares and minimum-norm solutions
- Complex symmetric matrices
- Inertia of normal matrices
- Eigenvalue and singular-value interlacing
- Inequalities among eigenvalues, singular values, and diagonal entries

Structure of the Book

A comprehensive list of symbols and notation (with page references) follows the Preface.

Chapter 1 reviews complex and real vector spaces, with numerous examples. The essential concepts of linear independence, linear dependence, and spanning lists are introduced, and the book's first matrix factorization emerges: the pivot-column decomposition.

Chapter 2 focuses on bases, dimension, and change-of-basis matrices. Matrix similarity arises as the relation between the representations of a linear transformation with respect to two bases. Lagrange interpolation provides examples of bases in vector spaces of polynomials. We observe instability with interpolation at equally spaced nodes and better behavior with interpolation at Chebyshev nodes. Integration of a polynomial interpolation leads to Simpson's rule and other Newton–Cotes quadrature formulae.

The block-matrix paradigm used throughout the book is introduced in Chapter 3. Block-matrix notation is useful in thinking about and communicating mathematical concepts. It focuses attention on the main ideas, instead of a quagmire of symbols and subscripts. Block matrices are central to the logic and coding of modern numerical algorithms. Row and column partitions are essential to the representation of a matrix product as the sum of outer products. Block Gaussian elimination leads to the Schur complement and determinant formulae for bordered matrices. Kronecker (tensor) products are a special topic at the end of this chapter.

Rank is the core concept in Chapter 4, which begins with the rank-nullity theorem and the subspace-intersection theorem. Block-matrix methods and full-rank factorizations are employed to present the basic rank inequalities for matrix sums and products. We present an algorithm to obtain LU factorizations that makes use of the outer-product representation for a matrix product. Shoda's theorem about matrix commutators and trace-zero matrices is a special topic in the final section.

Chapters 5 and 6 review geometry in the Euclidean plane and use it to motivate axioms for inner product spaces and normed linear spaces. Topics include orthogonal vectors, orthogonal projections, orthonormal bases, orthogonalization, the Riesz representation theorem, adjoints, and applications of the theory to Fourier series. Orthogonal polynomials and Gaussian quadrature are the special topic in Chapter 6.

Chapter 7 introduces unitary matrices, which are used in modern computational algorithms because they are easy to invert and exhibit superior stability properties in numerical calculations. In this chapter, we use unitary matrices to construct the QR factorization and a unitary similarity to upper Hessenberg form.

Chapter 8 discusses orthogonal projections, best approximations, least-squares solutions of linear systems, and the use of QR factorizations to solve the normal equations.

Chapter 9 introduces eigenvalues, eigenvectors, and geometric multiplicity. We show that an $n \times n$ real or complex matrix has at least one and not more than n distinct eigenvalues, and use Gershgorin's disk theorem to identify a region in the complex plane that contains them.

Chapter 10 deals with the characteristic polynomial and algebraic multiplicity. We develop criteria for diagonalizability and define primary matrix functions of a diagonalizable matrix. Topics include Fibonacci numbers, the eigenvalues of AB and BA, commutants, and simultaneous diagonalization.

Chapter 11 features Schur's triangularization theorem and a related result for a commuting family. Schur's theorem is used to prove the Cayley–Hamilton theorem: each square matrix is annihilated by its characteristic polynomial. The latter result motivates introduction of the minimal polynomial and a study of its properties. We prove Sylvester's theorem on linear matrix equations and use it to show that every square matrix is similar to a block diagonal matrix with unispectral diagonal blocks. The special topic in this chapter discusses perturbations of the Google matrix that facilitate computation of website rankings.

Chapter 12 builds on the preceding chapter to show that every square matrix is similar to a Jordan matrix that is unique up to permutation of its direct summands.

We discuss several applications of the Jordan canonical form in Chapter 13. They include systems of linear differential equations, an analysis of the Jordan structures of AB and BA, convergent and power-bounded matrices, a limit theorem for stochastic matrices that have positive entries, similarity of a matrix to its transpose, and similarity of a matrix to its complex conjugate.

Chapter 14 is about normal matrices: matrices that commute with their conjugate transpose. The spectral theorem says that a matrix is normal if and only if it is unitarily diagonalizable. Hermitian, skew-Hermitian, unitary, real-orthogonal, real-symmetric, and circulant matrices are all normal.

Positive semidefinite matrices are the subject of Chapter 15. These matrices arise in statistics (correlation matrices and the normal equations), mechanics (kinetic and potential energy in a vibrating system), and geometry (ellipsoids). Topics include matrix square roots, simultaneous diagonalization of quadratic forms, Cholesky factorization, and Hadamard and Kronecker products.

The principal result in Chapter 16 is the singular value decomposition, which is at the heart of many modern numerical algorithms in statistics, control theory, approximation, image compression, and data analysis. Topics include the compact singular value decomposition and polar decompositions, with special attention to uniqueness of these factorizations. A special topic is unitary equivalence of a complex matrix to an upper bidiagonal matrix.

In Chapter 17, the singular value decomposition is used to compress an image or data matrix. Other applications of the singular value decomposition discussed are the generalized inverse (Moore–Penrose inverse) of a matrix; inequalities between singular values and eigenvalues; the spectral norm of a matrix; perturbation bounds for linear systems and eigenvalue problems; and canonical forms for matrices that are complex symmetric or idempotent.

Chapter 18 investigates eigenvalue interlacing phenomena for Hermitian matrices that are bordered or are subjected to an additive perturbation. Related results include an interlacing theorem for singular values, a determinant criterion for positive definiteness, and inequalities that link the eigenvalues and diagonal entries of a Hermitian matrix. We prove Sylvester's inertia theorem for Hermitian matrices and a generalized inertia theorem for normal matrices.

Norms and matrix norms are the topics in Chapter 19. Eight examples of matrix norms are presented, with a systematic account of inequalities between pairs of them. Many iterative algorithms require that a particular matrix has spectral radius less than 1, which would be the case if some matrix norm of that matrix is less than 1. We analyze two iterative algorithms: the point Jacobi method to solve a linear system and the power method to find a dominant eigenpair. Facts about matrix norms are used to prove Gelfand's formula for the spectral radius.

Chapter 20 is devoted to Perron's theorem about the dominant eigenpair and limits of powers of a real square matrix with positive entries. This result has been used in diverse fields such as economic modeling, team ranking, population dynamics, genetics, and city planning. We prove it using facts about matrix norms and the Jordan canonical form.

Four short appendices review notation and concepts for complex numbers, polynomials, basic linear algebra, and mathematical induction. The appendices are provided for reference, and readers can consult them as needed.

The appendices are followed by a short list of references and an extensive index.

Acknowledgments

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Notation

$\in, \notin$	is / is not an element of
$\subseteq$	is a subset of
$\varnothing$	the empty set
$\cup$	union
$\cap$	intersection
$\times$	Cartesian product
$f : X \to Y$	f is a function from X into Y
$\Longrightarrow$	implies
$\Longleftarrow$	is implied by
$\Longleftrightarrow$	if and only if
$\approx$	approximately equal
$x \mapsto y$	implicit definition of a function that maps x to y
$\mathbb{N} = \{1, 2, 3, \ldots\}$	the set of natural numbers
$\mathbb{Z} = \{\ldots, -2, -1, 0, 1, 2, \ldots\}$	the set of integers
$\mathbb{R}$	the set of real numbers
$\mathbb{C}$	the set of complex numbers
$\mathbb{F}$	field of scalars ($\mathbb{F} = \mathbb{R}$ or $\mathbb{C}$)
$\operatorname{Re} z$	real part of the complex number z (p. 417)
$\operatorname{Im} z$	imaginary part of the complex number z (p. 417)
$\lvert z \rvert$	modulus of the complex number z (p. 420)
$\arg z$	argument of the complex number z (p. 420)
$[a, b]$	a real interval that includes its endpoints a, b
$\mathcal{U}, \mathcal{V}, \mathcal{W}$	vector spaces
$\mathscr{U}, \mathscr{V}, \mathscr{W}$	subsets of vector spaces
$a, b, c, \ldots$	scalars
$\mathbf{a}, \mathbf{b}, \mathbf{c}, \ldots$	(column) vectors
$A, B, C, \ldots$	matrices
$\mathsf{M}_{m\times n}(\mathbb{F})$	the set of $m \times n$ matrices with entries in $\mathbb{F}$
$\mathsf{M}_n(\mathbb{F})$	the set of $n \times n$ matrices with entries in $\mathbb{F}$
$\mathsf{M}_{m\times n}$	the set of $m \times n$ matrices with entries in $\mathbb{C}$
M_n	the set of $n \times n$ matrices with entries in $\mathbb{C}$
$\cong$	an equivalence relation (p. 51)
$\deg p$	degree of a polynomial p (p. 428)
δ_{ij}	Kronecker delta (p. 434)
$(\mathbf{x})_i$	ith entry of a vector $\mathbf{x}$ (p. 439)
I_n	$n \times n$ identity matrix (p. 434)

I	identity matrix (size inferred from context) (p. 434)
$\operatorname{diag}(x_1, x_2, \ldots, x_n)$	diagonal matrix with diagonal entries $x_1, x_2, \ldots, x_n$ (p. 435)
$A^0 = I$	convention for zeroth power of a matrix (p. 437)
A^T	transpose of A (p. 437)
$A^{-\mathsf{T}}$	inverse of A^T (p. 437)
$\overline{A}$	conjugate of A (p. 437)
A^*	conjugate transpose (adjoint) of A (p. 437)
A^{-*}	inverse of A^* (p. 437)
$\operatorname{tr} A$	trace of A (p. 438)
$\det A$	determinant of A (p. 441)
$\operatorname{adj} A$	adjugate of A (p. 442)
$\operatorname{sgn} \sigma$	sign of a permutation σ (p. 443)
$\mathcal{P}_n$	set of complex polynomials of degree at most n (p. 4)
$\mathcal{P}_n(\mathbb{R})$	set of real polynomials of degree at most n (p. 4)
$\mathcal{P}$	set of all complex polynomials (p. 4)
$C_\mathbb{F}[a,b]$	set of continuous $\mathbb{F}$-valued functions on $[a,b]$ (p. 4)
$C[a,b]$	set of continuous $\mathbb{C}$-valued functions on $[a,b]$ (p. 4)
$\operatorname{null} A$	null space of a matrix A (p. 5)
$\operatorname{col} A$	column space of a matrix A (p. 6)
$\operatorname{row} A$	row space of a matrix A (p. 6)
$\mathcal{P}_{\text{even}}$	set of even complex polynomials (p. 7)
$\mathcal{P}_{\text{odd}}$	set of odd complex polynomials (p. 7)
$A\mathcal{U}$	A acting on a subspace $\mathcal{U}$ (p. 6)
$\operatorname{span} \mathscr{S}$	span of a subset $\mathscr{S}$ of a vector space (p. 8)
$\mathbf{e}$	all-ones vector (p. 10)
$\mathcal{U} \cap \mathcal{W}$	intersection of subspaces $\mathcal{U}$ and $\mathcal{W}$ (p. 11)
$\mathcal{U} + \mathcal{W}$	sum of subspaces $\mathcal{U}$ and $\mathcal{W}$ (p. 12)
$\mathcal{U} \oplus \mathcal{W}$	direct sum of subspaces $\mathcal{U}$ and $\mathcal{W}$ (p. 12)
$\mathbf{v}_1, \mathbf{v}_2, \ldots, \widehat{\mathbf{v}_j}, \ldots, \mathbf{v}_r$	list of vectors with $\mathbf{v}_j$ omitted (p. 17)
$\mathbf{e}_1, \mathbf{e}_2, \ldots, \mathbf{e}_n$	standard basis for $\mathbb{F}^n$ (p. 28)
E_{ij}	matrix with (i,j) entry 1 and all others 0 (p. 28)
$\dim \mathcal{V}$	dimension of $\mathcal{V}$ (p. 28)
$[\mathbf{v}]_\beta$	coordinate vector of $\mathbf{v}$ with respect to a basis β (p. 33)
$\mathfrak{L}(\mathcal{V}, \mathcal{W})$	set of linear transformations from $\mathcal{V}$ to $\mathcal{W}$ (p. 35)
$\mathfrak{L}(\mathcal{V})$	set of linear transformations from $\mathcal{V}$ to itself (p. 35)
$\ker T$	kernel of T (p. 36)
$\operatorname{ran} T$	range of T (p. 36)
I	identity linear transformation (p. 38)
$\operatorname{rank} A$	rank of a matrix A (p. 31)
$\operatorname{nullity} A$	nullity of a matrix A (p. 74)
$\star$	unspecified matrix entry (p. 63)
$A \oplus B$	direct sum of matrices A and B (p. 64)
$A \otimes B$	Kronecker product of matrices A and B (p. 68)
$\operatorname{vec} A$	vector of stacked columns of a matrix A (p. 69)

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$[A, B]$	commutator of matrices A and B (p. 85)
$\langle \mathbf{x}, \mathbf{y} \rangle$	inner product of vectors $\mathbf{x}$ and $\mathbf{y}$ (p. 98)
$\langle A, B \rangle_{\mathrm{F}}$	Frobenius inner product of matrices A and B (p. 99)
$\perp$	orthogonal (p. 100)
$\lVert \mathbf{x} \rVert$	norm of a vector $\mathbf{x}$ (p. 101)
$\lVert \mathbf{x} \rVert_2$	Euclidean norm of a vector $\mathbf{x}$ (p. 101)
$\lVert A \rVert_{\mathrm{F}}$	Frobenius norm of a matrix A (p. 101)
$\lVert \mathbf{x} \rVert_1$	ℓ^1 norm (absolute sum norm) of a vector $\mathbf{x}$ (p. 106)
$\lVert \mathbf{x} \rVert_\infty$	ℓ^∞ norm (max norm) of a vector $\mathbf{x}$ (p. 106)
${}_\gamma[T]_\beta$	matrix representation of $T \in \mathfrak{L}(\mathcal{V}, \mathcal{W})$ with respect to bases β and γ (p. 118)
$f(x^+)$	one-sided limit from the right (p. 126)
$f(x^-)$	one-sided limit from the left (p. 126)
$P_{\mathbf{v}}$	projection onto a unit vector $\mathbf{v}$ (p. 138)
F_n	$n \times n$ Fourier matrix (p. 140)
$\mathscr{U}^\perp$	orthogonal complement of a set $\mathscr{U}$ (p. 158)
$P_{\mathcal{U}}$	orthogonal projection onto a subspace $\mathcal{U}$ (p. 163)
$d(\mathbf{v}, \mathcal{U})$	distance from $\mathbf{v}$ to $\mathcal{U}$ (p. 167)
$G(\mathbf{u}_1, \mathbf{u}_2, \ldots, \mathbf{u}_n)$	Gram matrix (p. 170)
$g(\mathbf{u}_1, \mathbf{u}_2, \ldots, \mathbf{u}_n)$	Gram determinant (p. 170)
$\operatorname{spec} A$	spectrum of A (p. 186)
$\mathcal{E}_\lambda(A)$	eigenspace of A for eigenvalue λ (p. 189)
$\mathscr{G}_k(A)$	kth Gershgorin disk of A (p. 190)
$\mathscr{G}(A)$	Gershgorin region of A (p. 190)
$R'_k(A)$	kth deleted absolute row sum of A (p. 190)
$R_k(A)$	kth absolute row sum of A (p. 190)
$p_A(z)$	characteristic polynomial of A (p. 202)
$\mathscr{F}'$	commutant of a set of matrices $\mathscr{F}$ (p. 213)
e^A	matrix exponential of A (p. 213)
$m_A(z)$	minimal polynomial of A (p. 226)
C_p	companion matrix of the polynomial p (p. 227)
$J_k(\lambda)$	$k \times k$ Jordan block with eigenvalue λ (p. 241)
J_k	$k \times k$ nilpotent Jordan block (p. 241)
$w_1, w_1, \ldots, w_q$	Weyr characteristic of a matrix (p. 248)
$\rho(A)$	spectral radius of A (p. 259)
$\mathcal{G}_\lambda(A)$	generalized eigenspace of A for eigenvalue λ (p. 253)
$\Delta(A)$	defect from normality of A (p. 282)
H_n	$n \times n$ Hartley matrix (p. 286)
R_n	real part of F_n (p. 286)
T_n	imaginary part of F_n (p. 286)
$A \circ B$	Hadamard product of A and B (p. 315)
$A \bullet B$	Jordan product of A and B (p. 320)
$\sigma_{\max}(A)$	maximum singular value (p. 346)
$\lVert A \rVert_2$	spectral norm of a matrix A (p. 347)
$\sigma_{\min}(A)$	minimum singular value (p. 350)
$\sigma_1(A), \sigma_2(A), \ldots$	singular values of A (p. 350)

$A^\dagger$	pseudoinverse of A (p. 353)
$\kappa_2(A)$	spectral condition number of A (p. 358)
$\mathscr{W}(A)$	numerical range of A (p. 384)
$\\|\mathbf{x}\\|_{[k]}$	k-norm of a vector $\mathbf{x}$ (p. 391)
$\\|\mathbf{x}\\|_S$	$\\|S\mathbf{x}\\|$, in which S is invertible (p. 391)
$\\|A\\|_1$	ℓ_1 norm of a matrix A (p. 391)
$\\|A\\|_\infty$	max norm of a matrix A (p. 392)
$N_{n\text{-max}}(A)$	n-max norm of a matrix A (p. 392)
$N_{\text{col max}}(A)$	column max norm of a matrix A (p. 392)
$N_S(A)$	$\\|SAS^{-1}\\|$ (p. 393)
$N_1(A)$	maximum absolute column-sum norm (p. 394)
$N_\infty(A)$	maximum absolute row-sum norm (p. 395)
$A_{(k)}$	$k \times k$ leading principal submatrix of A (p. 436)
S_n	nth statement in an induction (p. 447)
$\lvert A\rvert$	modulus of A (p. 334)
$\lvert A\rvert$	$A = [a_{ij}] \implies \lvert A\rvert = [\lvert a_{ij}\rvert]$ (Chapter 20 only)
$\lvert\mathbf{x}\rvert$	$\mathbf{x} = [x_i] \implies \lvert\mathbf{x}\rvert = [\lvert x_i\rvert]$ (Chapter 20 only)