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SINGULARLY PERTURBED METHODS FOR NONLINEAR ELLIPTIC PROBLEMS

This introduction to singularly perturbed methods in nonlinear elliptic partial differential equations emphasises the existence and local uniqueness of solutions exhibiting a concentration property. The authors avoid using sophisticated estimates and explain the main techniques by thoroughly investigating two relatively simple but typical non-compact elliptic problems. Each chapter then progresses to other related problems to help the reader learn more about the general theories developed from singularly perturbed methods.

Designed for PhD students and junior mathematicians intending to perform research in the area of elliptic differential equations, the text covers three main topics. The first is the compactness of the minimization sequences, or the Palais–Smale sequences, or a sequence of approximate solutions; the second is the construction of peak or bubbling solutions by using the Lyapunov–Schmidt reduction method; and the third is the local uniqueness of these solutions.

Daomin Cao is a professor at the Institute of Applied Mathematics, Chinese Academy of Sciences. His research focuses on nonlinear partial differential equations. He was awarded the first-class prize for Outstanding Young Scientists by the Chinese Academy of Sciences. He is the editor of academic mathematical journals, including *Applicable Analysis, Annales Academiac Scientiarum Fennicae Mathematica*, and *Acta Mathematicae Applicatae Sinica*.

Shuangjie Peng is a professor at the School of Mathematics and Statistics, Central China Normal University. His research focuses on nonlinear elliptic problems. He was awarded the first-class prize in Natural Sciences by the Hubei Province and the second-class prize in Natural Sciences by the Ministry of Education. He is the editor of several academic mathematical journals, including *Communications on Pure and Applied Analysis*, *Acta Mathematica Scientia*, and *Acta Mathematicae Applicatae Sinica*.

Shusen Yan is a professor at the School of Mathematics and Statistics, Central China Normal University. His main research interests are nonlinear elliptic problems.

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Singularly Perturbed Methods for Nonlinear Elliptic Problems

DAOMIN CAO
Chinese Academy of Sciences

SHUANGJIE PENG
Central China Normal University

SHUSEN YAN
Central China Normal University

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Preface

The development of variational methods is centered around a fundamental goal – namely, to find solutions for partial differential equations with variational structure. The history of such methods dates back to the nineteenth century. Despite their long history, these methods still have a strong impact on today's research. New critical point theories, such as the mountain pass lemma of Ambrosetti and Rabinowitz and the concentration compactness principle of P. L. Lions (see also Aubin [10], Brezis and Nirenberg [24]), have led to many remarkable results that verify the existence of nontrivial solutions for superlinear elliptic problems. However, these results are not very effective for finding solutions with higher energy for non-compact elliptic problems.

Fortunately, this dilemma was resolved thanks to the celebrated works of Bahri [12], Rey [125] and W.-M. Ni-Takagi [113] in the late 1980s, which opened the door to the construction of solutions for elliptic problems. In the last three decades, many powerful techniques based on the classical Lyapunov–Schmidt reduction procedure have been developed in the study of the existence of solutions for nonlinear elliptic problems and the study of these solutions' properties. However, the discussion of these techniques has been spread out among various articles, most of which are considered too technical for PhD students or even junior working mathematicians to read. This book aims to explain the main ideas associated with these techniques in a self-contained manner by investigating two typical non-compact elliptic problems. The book consists of five chapters and an appendix that focuses on the analytical aspects of nonlinear elliptic problems.

Chapter 1 is devoted to studying the existence of least-energy solutions for some typical nonlinear elliptic problems. The emphasis here is on the Schrödinger equations in $\mathbb{R}^N$ with subcritical growth and semilinear elliptic problems involving a Sobolev critical exponent in bounded domains of $\mathbb{R}^N$.

One of the main topics of the discussion is how the compactness of the minimization sequences(or the so-called Palais-Smale sequences) can be recovered by an energy constraint. We also discuss a global compactness result, showing how a Palais–Smale sequence may lose its compactness. This chapter serves as a preliminary for the topics discussed in the subsequent chapters of this book.

The Lyapunov–Schmidt reduction method and its variants have been widely used to construct peak solutions or bubbling solutions for singularly perturbed elliptic problems. Such solutions concentrate on a finite number of points. We will discuss three main issues: the necessary condition for the location of the concentration points for the peak solutions or the bubbling solutions, the existence of such solutions, and the local uniqueness of such solutions. The first two issues will be discussed in Chapter 2, while the third one is studied in Chapter 3.

The Lyapunov–Schmidt reduction argument is an application of the implicit function theorem. The difficulty in carrying out this argument is that the corresponding linear operator for the approximate solution is not invertible on the whole space. In view of this difficulty, determining the approximate kernel for this linear operator becomes essential. To illustrate this idea, we study two typical singularly perturbed elliptic problems in Chapter 2: the nonlinear Schrödinger equations in $\mathbb{R}^N$ with subcritical growth and the Brezis–Nirenberg problem. These examples were chosen so that numerous sophisticated estimates can be avoided.

Local Pohozaev identities are used to obtain the necessary condition for the location of the peak solutions and the bubbling solutions in Chapter 2. In Chapter 3, they are used to study the local uniqueness of such solutions. The advantages of using the local Pohozaev identities to study the local uniqueness problems are twofold. Firstly, the classical degree-counting methods rely heavily on the estimates of second-order derivatives of the solutions, whereas the arguments via the local Pohozaev identities only involve the estimates of the first-order derivatives of the solutions, which simplifies the problem. Secondly, the methods via local Pohozaev identities can be adapted to study other problems, where the classical degree-counting methods do not work.

The Lyapunov–Schmidt reduction method and its variants are typically used to construct solutions for elliptic problems with small parameters. In Chapter 4, these methods are adapted to study some non-singularly perturbed elliptic problems. Again, to avoid complicated estimates that may obfuscate the main ideas, we only study the Schrödinger equations with subcritical growth. The main idea in employing a reduction argument for non-singularly perturbed elliptic problems is that we can use a large integer k as the parameter in the construction of solutions, where k is the number of the bumps or the number of bubbles in the approximate solutions. The results in Chapter 4 show that the non-compactness of some elliptic problems may give rise to the existence of infinitely many positive solutions, whose energy can be arbitrarily large. We remark that such results cannot be obtained by using the abstract critical points theories.

The singularly perturbed problems studied in Chapter 2 have concentration solutions. On the other hand, it is also important to study the non-existence of concentration solutions for other singularly perturbed elliptic problems. Results in this direction are obtained by using a local Pohozaev identity, as presented in Chapter 5. As an application of these results, the existence of infinitely many sign-changing solutions for the Brezis-Nirenberg problem is proven.

For the convenience of our readers, we collect some of the results required for this book in the Appendix. These include some basic estimates, various Pohozaev identities, some preliminary properties of Sobolev spaces, certain fundamental estimates on elliptic equations, the Kelvin transformation, the kernel of some linear operators, and the estimate for the Green's function.

The Pohozaev identity was derived in the 1960s and was used in proving the non-existence of nontrivial solutions for some nonlinear elliptic problems. In this book, we emphasise the important role that various Pohozaev identities can also play in other problems, such as in the problems of the local uniqueness of solutions and the existence of solutions.

This book is written for those who have some knowledge of Sobolev spaces, nonlinear functional analysis and various estimates for elliptic equations. It aims to introduce some typical techniques for the construction of solutions to nonlinear elliptic problems. As this is not a survey article, many important results obtained in the last three decades have not been included. Also, it is hopelessly impossible to list all the relevant papers from the last three decades in the references; thus we will only briefly discuss other related problems and provide some references for those problems at the end of the monograph.

Portions of the materials contained herein were taken from the lecture notes prepared by the authors for the 2015 summer program for partial differential equations, chaired by Prof. Zhouping Xin. We would like to take this opportunity to thank Professor Xin for his support. We would also like to thank Prof. Meng Fai Lim for his help with the English presentation of this book. Last but not least, a prior version of this book served as a reference text for the course 'Perturbed Methods in Elliptic Equations' in Central China Normal University during the autumn semester of 2017. We would like to thank the graduate students of that course for their valuable feedback on this book.

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