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1
Non-compact Elliptic Problems

The solvability of elliptic equations with variational structure can be studied by detecting the existence of minimizers of the corresponding constrained minimization problems or critical points of the corresponding functionals. In this chapter, we will study the possible loss of compactness of some sequences of approximating critical points for some functionals. This will lay the foundation for the construction of peak or bubbling solutions in Chapters 2 and 4, and for the discussion of the multiplicity results in Chapter 5. To show the main ideas, we will consider two typical types of elliptic problems: the Schrödinger equation in $\mathbb{R}^N$ and the semilinear elliptic problems with a critical Sobolev exponent in a bounded domain of $\mathbb{R}^N$. One of the main topics in this chapter is how one can recover the compactness of the minimization sequences, or the Palais–Smale sequences, by an energy constraint. We will also discuss a global compactness result that shows how a Palais–Smale sequence may lose its compactness.

The theories in Chapter 1 were mainly developed in the 1970s and 1980s. The readers interested in other aspects of critical point theory can refer to the books [124, 134, 151].

1.1 Minimization Problems with Compactness

Consider the following eigenvalue problem

$$\begin{cases} -\Delta u = \lambda f(x, u), & \text{in } \Omega, \\ u = 0, & \text{on } \partial\Omega, \end{cases} \tag{1.1.1}$$

for some unknown pair (λ, u), where Ω is a bounded domain in $\mathbb{R}^N$ and $f(x,t) \in C(\Omega \times \mathbb{R})$ with the property that $|f(x,t)| \leq C(1 + |t|^p)$ for some constants $C > 0$ and $p \in \big(1, 2^* - 1\big)$. Here we denote 2^* to be $\frac{2N}{N-2}$ or $+\infty$ according as $N \geq 3$ or $N = 1, 2$.

Before solving the above eigenvalue problem, let us briefly discuss the following constraint minimization problem in calculus.

Let $g(x)$ and $G(x)$ be two C^1 functions in $\mathbb{R}^N$. Suppose that $\nabla G(x) \neq 0$ for any $x \in \mathbb{R}^N$ and that the equation $G(x) = 0$ defines a compact closed hyper-surface in $\mathbb{R}^N$. Define

$$m := \inf\{g(x) : x \in \mathbb{R}^N,\ G(x) = 0\}. \tag{1.1.2}$$

Theorem 1.1.1 *The quantity m defined in problem* (1.1.2) *is achieved by some* x_0. *Moreover, this* x_0 *satisfies*

$$\nabla g(x_0) = \lambda \nabla G(x_0), \tag{1.1.3}$$

for some $\lambda \in \mathbb{R}$.

Proof. Step 1. Take a minimization sequence $\{x_n\}$ such that $G(x_n) = 0$ and

$$m \leq g(x_n) \leq m + \frac{1}{n}. \tag{1.1.4}$$

Step 2. Since the set $\{x : G(x) = 0\}$ is compact, there exists a subsequence which, by abuse of notation, is still denoted by $\{x_n\}$, and x_0 such that $x_n \to x_0$. Since $G(x_n) = 0$, we have that $G(x_0) = 0$.

Step 3. It follows from (1.1.4) and the continuity of g that

$$m \geq \lim_{n\to\infty} g(x_n) = g(x_0) \geq m.$$

This shows that m is attainable.

Step 4. It remains to prove (1.1.3). For any tangent vector $\vec{l}$ of the hyper-surface $\{x : G(x) = 0\}$ at x_0, we take a curve $x(t)$ on $\{x : G(x) = 0\}$ such that

$$\begin{cases} x'(t)|_{t=0} = \vec{l}, \\ x(0) = x_0. \end{cases}$$

Let $h(t) := g(x(t))$. Then $h(0) = g(x_0)$ and $h(t)$ achieves its minimum at $t = 0$. Therefore,

$$\langle \nabla g(x_0), \vec{l} \rangle = h'(0) = 0.$$

This implies that $\nabla g(x_0)$ is perpendicular to any tangent vector of the surface $\{x : G(x) = 0\}$. In other words, $\nabla g(x_0)$ is a normal vector of this surface $G(x) \equiv 0$ at x_0. On the other hand, for any $z \in \{x : G(x) = 0\}$, $\nabla G(z)$ is a normal vector of the surface $\{x : G(x) = 0\}$ at z. So we find that at the minimum point x_0, ∇G and ∇g are parallel, and therefore, there exists $\lambda \in \mathbb{R}$ such that

$$\nabla g(x_0) = \lambda \nabla G(x_0). \qquad \square$$

Remark 1.1.2 We note that the proof of the existence of a minimizer for the constraint minimization problem (1.1.2) only requires the following assumptions: (i). The compactness of the set $\{x : G(x) = 0\}$. (ii). $g(x)$ is lower semi-continuous, i.e., $\liminf_{x \to x_0} g(x) \geq g(x_0)$.

In the sequel, we define the space $H_0^1(\Omega)$ to be the closure of $C_0^\infty(\Omega)$ with respect to the norm $\|u\| = (\int_\Omega |\nabla u|^2)^{\frac{1}{2}}$, which is induced by the inner product $\langle u, v\rangle = \int_\Omega \nabla u \nabla v$. For some important results on the space $H_0^1(\Omega)$, the readers can refer to the appendix.

Let us now study the eigenvalue problem (1.1.1). The weak solution $u \in H_0^1(\Omega)$ of (1.1.1) satisfies the following relation:

$$\int_\Omega \nabla u \nabla \varphi = \lambda \int_\Omega f(x, u)\varphi, \quad \forall\, \varphi \in H_0^1(\Omega). \tag{1.1.5}$$

It is well known that the left-hand side of (1.1.5) is the derivative of the functional

$$I(u) = \frac{1}{2} \int_\Omega |\nabla u|^2, \quad u \in H_0^1(\Omega),$$

while the right-hand side is the derivative of the functional $\int_\Omega F(x, u)$, where $F(x, t) = \int_0^t f(x, s)\, ds$. Thus, a weak solution (λ, u) of (1.1.1) satisfies

$$\nabla I(u) = \lambda \nabla \int_\Omega F(x, u).$$

Hence we are led to study a problem similar to the algebraic equation (1.1.3). In view of this, it is then natural to consider the following constraint minimization problem:

$$m := \inf \left\{ I(u) : u \in H_0^1(\Omega), \int_\Omega F(x, u) = 1 \right\}. \tag{1.1.6}$$

To prove that problem (1.1.6) is achieved by some $u \in H_0^1(\Omega)$, we first take a minimization sequence $\{u_n\} \subset H_0^1(\Omega)$, satisfying

$$m \leq I(u_n) \leq m + \frac{1}{n}, \quad \int_\Omega F(x, u_n) = 1. \tag{1.1.7}$$

From the proof of Theorem 1.1.1, we see that we need to prove the following:

(i) There is a subsequence of $\{u_n\}$, still denoted by $\{u_n\}$, such that u_n converges to u in 'certain' sense.
(ii) The functional $I(u)$ is lower semi-continuous. That is, $\lim_{n\to+\infty} I(u_n) \geq I(u)$.
(iii) The functional $\int_\Omega F(x,u)$ is continuous. That is, $\lim_{n\to+\infty} \int_\Omega F(x,u_n) = \int_\Omega F(x,u)$.

Remark 1.1.3 (1) $H_0^1(\Omega)$ is generally not compact. However, we do have that $H_0^1(\Omega)$ is weakly compact (see Theorem 6.3.1). In other words, any bounded sequence $\{u_n\}$ in $H_0^1(\Omega)$ has a subsequence, which we still denote by $\{u_n\}$, such that for any $\varphi \in H_0^1(\Omega)$, one has $\langle u_n, \varphi\rangle \to \langle u, \varphi\rangle$ for some $u \in H_0^1(\Omega)$. We use the notation $u_n \rightharpoonup u$ to denote the weak convergence of u_n to u in $H_0^1(\Omega)$.

(2) $I(u)$ is weakly lower semi-continuous in $H_0^1(\Omega)$. Consequently, if $u_n \rightharpoonup u$ in $H_0^1(\Omega)$, then

$$\liminf_{n\to+\infty} I(u_n) \geq I(u).$$

(3) $H_0^1(\Omega) \hookrightarrow L^{p+1}(\Omega)$ is compact if $1 \leq p < \frac{N+2}{N-2}$ and Ω is bounded.

Theorem 1.1.4 *Suppose that* $\{u : u \in H_0^1(\Omega), \int_\Omega F(x,u) = 1\}$ *is not empty. Then* (1.1.6) *is achieved. Moreover, if* $\nabla \int_\Omega F(x,u) \neq 0$ *at* u, (1.1.5) *holds for some* $\lambda \in \mathbb{R}$.

Proof. Let $\{u_n\}$ be a minimization sequence in $H_0^1(\Omega)$ that satisfies (1.1.7). Then $\{u_n\}$ is bounded in $H_0^1(\Omega)$. By Theorems 6.3.1 and 6.3.7, there exists a subsequence of $\{u_n\}$, which one still denotes by $\{u_n\}$, and a function $u \in H_0^1(\Omega)$ such that

$$u_n \rightharpoonup u \qquad \text{weakly in } H_0^1(\Omega),$$

and

$$u_n \to u \qquad \text{strongly in } L^{p+1}(\Omega).$$

Recalling that $|f(x,t)| \leq C(1+|t|^p)$, we now see that there exists $\theta \in [0,\ 1]$,

$$\begin{aligned}
&\int_\Omega |F(x,u_n) - F(x,u)| \\
&= \left|\int_\Omega f(x, u+\theta(u_n-u))(u_n-u)\right| \\
&\leq \left(\int_\Omega |f(x,u+\theta(u_n-u))|^{\frac{p+1}{p}}\right)^{\frac{p}{p+1}} \left(\int_\Omega |u_n-u|^{p+1}\right)^{\frac{1}{p+1}}
\end{aligned}$$

$$\le C\Big(\int_\Omega (1+|u_n|^p+|u|^p)^{\frac{p+1}{p}}\Big)^{\frac{p}{p+1}}\Big(\int_\Omega |u_n-u|^{p+1}\Big)^{\frac{1}{p+1}}$$

$$\le C\Big(\int_\Omega |u_n-u|^{p+1}\Big)^{\frac{1}{p+1}} \to 0,$$

as $n\to\infty$.

On the other hand, we have

$$I(u_n)=\frac{1}{2}\int_\Omega|\nabla u|^2-\int_\Omega\nabla(u_n-u)\nabla u+\frac{1}{2}\int_\Omega|\nabla(u-u_n)|^2$$

$$\ge\frac{1}{2}\int_\Omega|\nabla u|^2-\int_\Omega\nabla(u_n-u)\nabla u$$

$$=I(u)+o(1),$$

where $o(1)\to 0$ as $n\to+\infty$.

Therefore, $m=\lim_{n\to\infty} I(u_n)\ge I(u)\ge m$. As a result, u achieves m.

Next, we prove (1.1.5). For any function $\varphi\in H_0^1(\Omega)$, $u+t\varphi$ may not satisfy $\int_\Omega F(x,u+t\varphi)=1$ for $t\neq 0$. Therefore, we need to correct this perturbation suitably. From the assumption $\nabla\int_\Omega F(x,u)\neq 0$, we can find a function $\psi\in H_0^1(\Omega)$ such that

$$\int_\Omega f(x,u)\psi\neq 0.$$

Define the following function

$$H(t,s)=\int_\Omega F(x,u+t\varphi+s\psi)-1.$$

Then one has

$$H(0,0)=0,\qquad \frac{\partial H(0,0)}{\partial s}=\int_\Omega f(x,u)\psi\neq 0.$$

By the implicit function theorem, there is a C^1 function $s(t)$ defined for small t such that $H(t,s(t))=0$, $s(0)=0$, and

$$s'(0)=-\frac{\displaystyle\int_\Omega f(x,u)\varphi}{\displaystyle\int_\Omega f(x,u)\psi}.$$

Now the function $\xi(t) = I\big(u + t\varphi + s(t)\psi\big)$ attains its minimum at $t = 0$. Hence $\xi'(0) = 0$. On the other hand, a direct calculation gives

$$\xi'(0) = \int_\Omega \nabla u \nabla \varphi - \lambda \int_\Omega f(x, u)\varphi,$$

where

$$\lambda = \frac{\displaystyle\int_\Omega \nabla u \nabla \psi}{\displaystyle\int_\Omega f(x, u)\psi}.$$

Hence the result follows. □

Remark 1.1.5 If $f(x, t) = |t|^{p-1}t$, we have $F(t) = \frac{1}{p+1}|t|^{p+1}$. In this simple case, for any $\varphi \in H_0^1(\Omega)$ and $|t|$ small, $\frac{(p+1)^{\frac{1}{p+1}}(u+t\varphi)}{|u+t\varphi|_{L^{p+1}(\Omega)}}$ satisfies the constraint.

A direct consequence of Theorem 1.1.4 is the existence of a positive solution for a nonlinear elliptic problem.

Theorem 1.1.6 *The following elliptic problem has a solution*

$$\begin{cases} -\Delta u = u^p, \ u > 0, & \text{in } \Omega, \\ u = 0, & \text{on } \partial\Omega, \end{cases} \tag{1.1.8}$$

where Ω is a bounded domain in $\mathbb{R}^N$ and $p \in (1, 2^* - 1)$.

Proof. Since $F(t) = \frac{1}{p+1}|t|^{p+1}$, we can always choose a minimization sequence, which is non-negative. It then follows from Theorem 1.1.4 that there exists a constant $\lambda \in \mathbb{R}$, $u \in H_0^1(\Omega)$ and $u \geq 0$ such that

$$-\Delta u = \lambda u^p, \quad \text{in } \Omega. \tag{1.1.9}$$

Since $\frac{1}{p+1}\int_\Omega |u|^{p+1} = 1$, we must have $u \not\equiv 0$. On the other hand, it is easy to see from (1.1.9) that

$$\lambda = \frac{\displaystyle\int_\Omega |\nabla u|^2}{\displaystyle\int_\Omega |u|^{p+1}} > 0.$$

By the maximum principle, we see that $u > 0$ in Ω. Hence $\lambda^{\frac{1}{p-1}}u$ is a solution of (1.1.8). □

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1.2 Non-compact Minimization Problems: The Concentration Compactness Principle

The minimization problem discussed in the previous section is compact in the sense that the functional $\int_\Omega F(x,u)$ is continuous under the weak convergence in $H_0^1(\Omega)$. Such compactness condition is not true anymore if Ω is unbounded, or $f(x,t)$ grows at the Sobolev critical rate of $|t|^{2^*-1}$ as $|t| \to +\infty$. In this section, we will consider two typical cases, where the compactness condition is violated.

Question 1. Suppose that $V(x) \in C(\mathbb{R}^N)$ satisfies

$$0 < V_0 \le V(x) \le V_1,$$

for some constants $V_1 > V_0 > 0$. We consider the following minimization problem

$$\inf\left\{\frac{1}{2}\int_{\mathbb{R}^N}(|\nabla u|^2 + V(x)|u|^2) : u \in H^1(\mathbb{R}^N),\ \int_{\mathbb{R}^N}|u|^{p+1} = 1\right\}, \tag{1.2.1}$$

where $1 < p < \frac{N+2}{N-2}$ if $N \ge 3$, $1 < p < +\infty$ if $N = 2$, and the space $H^1(\mathbb{R}^N)$ is the closure of $C_0^\infty(\mathbb{R}^N)$ under the norm $\|u\| = (\int_{\mathbb{R}^N}(|\nabla u|^2 + |u|^2))^{\frac{1}{2}}$, which is induced by the inner product $\langle u,\ v\rangle = \int_{\mathbb{R}^N}(\nabla u \nabla v + uv)$. We point out that, in this case, the functional $\frac{1}{2}\int_{\mathbb{R}^N}(|\nabla u|^2 + V(x)|u|^2)$ is weakly lower semi-continuous in $H^1(\mathbb{R}^N)$, but the functional $\int_{\mathbb{R}^N}|u|^{p+1}$ is not continuous under the weak convergence in $H^1(\mathbb{R}^N)$.

Question 2. Suppose that Ω is a bounded domain in $\mathbb{R}^N$, $N \ge 3$. Consider

$$\inf\left\{\frac{1}{2}\int_\Omega\left(|\nabla u|^2 - \lambda|u|^2\right) : u \in H_0^1(\Omega),\ \int_\Omega |u|^{2^*} = 1\right\}, \tag{1.2.2}$$

where $\lambda \in (0, \lambda_1)$. Here, we use λ_1 to denote the first eigenvalue of the operator $-\Delta$ in Ω, subject to the homogeneous Dirichlet boundary condition. We also emphasize that, in this case, $\frac{1}{2}\int_\Omega(|\nabla u|^2 - \lambda|u|^2)$ is weakly lower semi-continuous in $H_0^1(\Omega)$, but $\int_\Omega |u|^{2^*}$ is not continuous under the weak convergence in $H_0^1(\Omega)$.

Before we study the above minimization problems, let us first discuss a simple problem in calculus for some useful ideas. Let $f(x) \in C(\mathbb{R}^N)$ be bounded from below. Consider

$$m := \inf_{x\in\mathbb{R}^N} f(x). \tag{1.2.3}$$

A solution to such problem may not be achieved. For example, $\inf\limits_{x\in\mathbb{R}} \arctan x$ cannot be achieved.

Now we discuss the conditions which guarantee that the minimum in (1.2.3) is achieved. Choose a minimization sequence $\{x_n\} \subset \mathbb{R}^N$ such that $f(x_n) \to m$ as $n \to +\infty$. Then we have two possibilities:

(i) There exists a subsequence, which we still denote by $\{x_n\}$, such that $x_n \to x_0 \in \mathbb{R}^N$. Then $f(x)$ achieved m at x_0.

(ii) $|x_n| \to \infty$. In this case, we have

$$m = \lim_{n\to+\infty} f(x_n) \geq \liminf_{|x|\to+\infty} f(x) := m^\infty \geq m.$$

Thus, $m = m^\infty$.

The possible loss of compactness for a minimization sequence $\{x_n\}$ for (1.2.3) is when $|x_n| \to +\infty$. If this occurs, we have

$$\inf_{x\in\mathbb{R}^N} f(x) = \liminf_{|x|\to+\infty} f(x).$$

Thus, the loss of the compactness will not occur if the function $f(x)$ satisfies

$$\inf_{x\in\mathbb{R}^N} f(x) < \liminf_{|x|\to+\infty} f(x). \tag{1.2.4}$$

From the above discussion, we have the following theorem.

Theorem 1.2.1 *Suppose that $f(x)$ is bounded from below and is continuous (or lower semi-continuous). If* (1.2.4) *holds, then the minimum in* (1.2.3) *is achieved.*

Remark 1.2.2 $m^\infty := \liminf_{|x|\to+\infty} f(x)$ may be regarded as the limit problem of (1.2.3).

As seen from the above discussion, to study the minimization problems (1.2.1) and (1.2.2), we need to understand the possible loss of the compactness of the minimization sequence and determine the corresponding limit problem. The compactness of the minimization sequence can then be recovered by imposing a condition on the two problems.

1.2.1 A Minimization Problem in an Unbounded Domain

In this subsection, we will discuss (1.2.1). Let

$$I(u) = \int_{\mathbb{R}^N} (|\nabla u|^2 + V(x)u^2), \quad u \in H^1(\mathbb{R}^N),$$

where $V(x) \in C(\mathbb{R}^N)$ satisfies $0 < V_0 \le V(x) \le V_1$ for some constants $V_1 > V_0 > 0$, and

$$\lim_{|x|\to+\infty} V(x) = V_\infty > 0.$$

Consider the following constraint minimization problem

$$I_a := \inf\left\{ I(u) : u \in H^1(\mathbb{R}^N),\ \int_{\mathbb{R}^N} |u|^{p+1} = a > 0 \right\}. \tag{1.2.5}$$

To start, we summarize some properties for I_a. Using the change of variable $u = a^{\frac{1}{p+1}} v$, we can verify that

$$I_a = a^{\frac{2}{p+1}} I_1,$$

which gives

$$I_a < I_b + I_{a-b}, \quad \forall\, b \in (0, a). \tag{1.2.6}$$

It is easy to check that for any $a > 0$, one has $I_a > 0$. If I_a is achieved by some $u \in H^1(\mathbb{R}^N)$, then $|u|$ also achieves I_a. Hence we may assume that I_a is achieved by some non-negative function $u \not\equiv 0$. Moreover, it is easy to verify that u satisfies

$$\begin{cases} -\Delta u + V(x)u = \dfrac{I_a}{a} u^p,\ u > 0,\ x \in \mathbb{R}^N, \\ u \in H^1(\mathbb{R}^N). \end{cases} \tag{1.2.7}$$

Therefore, $w = (I_a/a)^{-\frac{1}{p-1}} u$ satisfies the following nonlinear Schrödinger equation

$$\begin{cases} -\Delta w + V(x)w = w^p,\ w > 0,\ x \in \mathbb{R}^N, \\ w \in H^1(\mathbb{R}^N). \end{cases} \tag{1.2.8}$$

To study whether (1.2.5) is achieved, we choose a minimizing sequence $\{u_n\} \subset H^1(\mathbb{R}^N)$ of I_a. Then $\{u_n\}$ is bounded in $H^1(\mathbb{R}^N)$. We may assume that there exists $u \in H^1(\mathbb{R}^N)$ such that

$$u_n \rightharpoonup u \text{ weakly in } H^1(\mathbb{R}^N),$$

and

$$u_n \to u \text{ strongly in } L^q_{loc}(\mathbb{R}^N) \text{ for any } q \in [2, 2^*).$$

From the weakly lower semi-continuity of $I(u)$ in $H^1(\mathbb{R}^N)$, we conclude that

$$I(u) \le \lim_{n\to+\infty} I(u_n) = I_a,$$

while Fatou's lemma implies that

$$\int_{\mathbb{R}^N} |u|^{p+1} \le \lim_{n\to\infty} \int_{\mathbb{R}^N} |u_n|^{p+1} = a.$$

Thus, in order to prove that u achieves I_a, we just need to prove that $\int_{\mathbb{R}^N} |u|^{p+1} = a$.

Following the same idea as in the discussion of (1.2.3), we need to exclude all the possibilities that may cause $\{u_n\}$ to lose compactness in $L^{p+1}(\mathbb{R}^N)$.

Let $v_n = u_n - u$. Then, for any $R > 0$, we have

$$\int_{B_R(0)} |v_n|^{p+1} \to 0, \quad \text{as } n \to +\infty.$$

Intuitively, if v_n does not converge strongly in $L^{p+1}(\mathbb{R}^N)$, we then have the following two cases.

Case 1. The main part of u_n lies in the infinity if $u \equiv 0$.

Case 2. u_n splits into two 'separate' parts u and v_n if $u \not\equiv 0$. This case is called the dichotomy case.

In case 1, we then have

$$\begin{aligned}\int_{\mathbb{R}^N} (V(x) - V_\infty)u_n^2 =& O\Big(\int_{B_R(0)} u_n^2\Big) + \int_{\mathbb{R}^N\setminus B_R(0)} (V(x) - V_\infty)u_n^2 \\ =& o(1) + O\left(\max_{|x|\ge R} |V(x) - V_\infty|\right) \int_{\mathbb{R}^N\setminus B_R(0)} u_n^2 \to 0,\end{aligned} \tag{1.2.9}$$

as $n \to +\infty$. This in turn implies that

$$I_a = \lim_{n\to+\infty} I(u_n) = \lim_{n\to+\infty} I^\infty(u_n) \ge I_a^\infty,$$

where

$$I_a^\infty = \inf\Big\{I^\infty(u) : u \in H^1(\mathbb{R}^N),\ \int_{\mathbb{R}^N} |u|^{p+1} = a\Big\}, \tag{1.2.10}$$

and

$$I^\infty(u) = \int_{\mathbb{R}^N} \big(|\nabla u|^2 + V_\infty u^2\big).$$

Similar to the discussion of (1.2.3), we call (1.2.10) the limit problem for (1.2.5). The above discussion shows that if $I_a < I_a^\infty$, then case 1 does not occur.

To exclude the dichotomy case, we need the following Brezis–Lieb lemma (see [23]).